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# Stein's method for models with general clocks: A tutorial

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Diffusion approximations are widely used in the analysis of service systems, providing tractable insights into complex models. While heavy-traffic limit theorems justify these approximations asymptotically, they do not quantify the error when the system is not in the limit regime. This paper presents a tutorial on the generator comparison approach of Stein's method for analyzing diffusion approximations in Markovian models where state transitions are governed by general clocks, which extends the well-established theory for continuous-time Markov chains and enables non-asymptotic error bounds for these approximations. Building on recent work that applies this method to single-clock systems, we develop a framework for handling models with multiple general clocks. Our approach is illustrated through canonical queueing systems, including the G/G/1 queue, the join-the-shortest-queue system, and the tandem queue. We highlight the role of the Palm inversion formula and the compensated queue-length process in extracting the diffusion generator. Most of our error terms depend only on the first three moments of the general clock distribution. The rest require deeper, model-specific, insight to bound, but could in theory also depend on only the first three moments.

*Key words*: Stein's method; diffusion approximation; steady-state; convergence rate

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# 1. Introduction

Fluid and diffusion approximations are indispensable in the study of service systems. Often depending only on the first two moments of the system's primitives, they can offer insights into the performance of an otherwise intractable model. Their use is commonly justified via heavy-traffic limit theorems, but these offer no insight into the quality of the approximation when the system is not in the asymptotic limit.

What *does* yield non-asymptotic approximation guarantees is a technique known as the generator comparison approach of Stein's method (Stein (1972), Barbour (1988, 1990), Götze (1991)). The generator approach has been widely applied to continuous-time Markov chains (CTMCs) and discrete-time Markov processes, but it was only recently extended by Braverman and Scully (2024) to continuous-time Markov processes where jumps are driven by clocks having general distributions—we call these general clocks. For example, the queue-length process in the $G/G/1$ system is driven by two general clocks tracking the residual interarrival and service times, while in the $M/M/1$ system, these clocks are exponentially distributed.

Service system models with general clocks are essential, since the assumption of exponentially distributed interarrival, service, or patience times is usually made for mathematical tractability instead of realism; e.g., Brown et al. (2005). Models are also known to exhibit qualitatively different behavior once the model primitives are no longer exponentially distributed; e.g., Whitt (1986), Bassamboo and Randhawa (2010), Dai et al. (2004). Furthermore, while it is possible to go beyond exponentially distributed service times using phase-type distributions, this complicates the model by increasing its dimensionality (Dai et al. (2010)).

The literature on heavy-traffic limits for models with general clocks is extensive (Whitt (2002)), but there is a gap in the theory when it comes to applying the generator approach to these models. A step towards bridging this gap was recently made by Braverman and Scully (2024), where the authors show how to apply the approach to models with a *single* general clock. This paper's goal is to provide an accessible tutorial on applying the approach to models with any number of general clocks.

The procedure for the applying the generator approach to models with exponential clocks has three main steps; for a tutorial, see Braverman et al. (2016). First, one begins with the infinitesimal generator of the CTMC and performs Taylor expansion to extract an approximating diffusion generator. Second, one uses the Poisson equation of the diffusion (or the CTMC as in Braverman (2022)) to write the distance between the CTMC and diffusion in terms of the difference of the two generators. Third, one uses a combination of Stein factor bounds (bounds on the derivatives of the Poisson equation solution) and moment bounds to bound this generator difference; some models may also require bounding a state-space collapse (SSC) term. Steps one and two are typically straightforward, and it is step three that requires the most effort and model-specific knowledge. The end result is a bound on the Wasserstein distance between the (scaled) pre-limit and limiting distributions.

We illustrate our approach using the queue-length processes of three canonical models: the queue-length process in the $G/G/1$ system, the join-the-shortest-queue (JSQ) system, and the tandem queue. The first model is the simplest, and we perform all three steps of the generator approach in full detail. The remaining models are more involved—the JSQ system has a SSC component and the tandem queue has a multidimensional diffusion approximation for which the Stein factor bounds are not known. Consequently, the corresponding diffusion approximation errors contain terms that require a deeper, model-specific analysis to bound; we leave this task to future work.

The main technical challenge to working with processes with general clocks is that there may not be a well-defined infinitesimal generator. Instead, we work with the stationary equation, also called basic adjoint relationship (BAR) by Braverman et al. (2017, 2024), and expand that to derive the diffusion generator. The BAR contains jump terms with respect to counting processes driven by the general clocks. The extra effort lies in extracting the diffusion generator from these jump terms. For this, we use the Palm inversion formula, which relates event averages to time averages.

In all three of our examples, instead of using the BAR for the queue length, we must work with the BAR for (what we call) the *compensated* queue length. The latter is a modification of the queue length, using residual interarrival and service times, so that the expected jump of the compensated process at arrival/departure times equals zero. Working with the compensated process eliminates the first-order Taylor expansion of the BAR jump terms; we discuss this in further detail at the end of Section 2.2.

### 1.1. Related work

The closest work to us is by Braverman and Scully (2024). They focus on carrying out steps one through three of the generator approach for the $G/G/1$ workload process, which is composed of the remaining workload in the system and a single general clock tracking the time until the next arrival. That paper develops both the classical generator approach (called the limiting approach there) and the more recently proposed prelimit approach, and much of the paper is focused on the Stein factor bounds for the workload process (step three). In contrast, the current paper focuses on illustrating step one of the (limiting) generator approach for a variety of models with multiple general clocks. It is worth noting that although Braverman and Scully (2024) also work with the compensated version of the workload process, they do so more out of algebraic convenience than necessity. In the three models we consider, using the compensated process appears necessary. It is also worth mentioning the recent work by Barbour et al. (2023), who develop theory for Gaussian-process approximation and apply it to several queueing models. They focus on process-level approximations, whereas we are focused on steady-state results.

Our work is conceptually similar to the recent line of work on the BAR approach by Braverman et al. (2017, 2024), Dai et al. (2025, 2024), Dai and Huo (2024b,a), which uses exponential test functions, carefully modified to eliminate the jump terms in the BAR and make it simpler to work with. In contrast, our test functions are not exponential. Our approach of using compensated processes eliminates only the first-order Taylor expansion terms of the BAR jump terms, and we have to carefully extract the diffusion generator from the second-order terms using the Palm inversion formula.

Studies of the JSQ system date back to Winston (1977) and Weber (1978). The discrete-time JSQ model has been analyzed using the drift method by Eryilmaz and Srikant (2012) and using Stein's method by Zhou and Shroff (2020); both papers rely critically on the assumption that the number of service completions in a timeslot is bounded. The continuous-time version with exponentially distributed interarrival and service times has been analyzed using Stein's method by Hurtado-Lange and Maguluri (2022) and Braverman (2023). Recently, Dai et al. (2024) used the BAR approach to establish a heavy-traffic limit theorem for the continuous-time JSQ model with general interarrival and service-time distributions; this is the same model as we consider in Section 3. Our diffusion approximation error in Proposition 2 contains a SSC term, and we suspect that the ideas of Dai et al. (2024) can be used as a starting point for bounding it.

The recent work of Dai et al. (2025) establishes a product-form heavy-traffic limit for generalized Jackson networks (GJNs) under a multi-scale heavy-traffic condition. This condition assumes that the spare capacity at station $j$ goes to zero at a rate of $r^j$, where $r$ is a heavy-traffic parameter; they also call this the load-separation condition. Surprisingly, their numerical results show that the product-form approximation performs well even when the load separation condition is violated and the parameters of the system correspond to the conventional heavy-traffic limit of Johnson (1983), Reiman (1984), which does not have a product-form distribution. A similar phenomenon is observed by Dai and Huo (2024a,b) for multiclass queueing networks with static buffer priority policies.

All three works rely on limit theorems, which cannot explain the surprising numerical accuracy of product-form approximations when the load-separation condition is violated. Since our tandem-queue in Section 4 is an example of a simple GJN, our BAR expansion offers a way to analyze the error of the product-form approximation of Dai et al. (2025).

## 2. The $G/G/1$ queue length

Consider a single-server queueing system operating under a first-come-first-serve discipline, where arrivals occur according to a renewal process. Let $U$ and $S$ be random variables having the interarrival and service time distributions, respectively and assume that

$$\mathbb{E}U^3 < \infty \quad \text{and} \quad \mathbb{E}S^3 < \infty.$$

For simplicity, we assume that simultaneous events (arrivals/departures) do not occur with probability one. Set

$$\lambda = 1/\mathbb{E}U, \quad \mu = 1/\mathbb{E}S, \quad \rho = \lambda/\mu,$$

$$c_U^2 = \lambda^2 \mathrm{Var}(U), \quad c_S^2 = \mu^2 \mathrm{Var}(S),$$

and assume that $\rho < 1$.

Let $Q(t)$ be the number of customers in system at time $t$ and set $X(t) = \delta Q(t)$, where $\delta = (1-\rho)$. Let $R_a(t)$ be the time until the next arrival. If $X(t) > 0$, let $R_s(t)$ be the remaining service time of the customer in service and, if $X(t) = 0$, let it denote the service time of the next customer to arrive. We are interested in the queue-length process

$$\{Z(t) = (X(t), R_a(t), R_s(t)) : t \geq 0\}.$$

We assume that the queue-length process is positive Harris recurrent (sufficient conditions are discussed by Dai and Meyn (1995)), and let $Z = (X, R_a, R_s)$ be the vector having its unique stationary distribution. Our assumption that $U$ and $S$ have finite third moments yields $\mathbb{E}X^2 < \infty$ (Theorem 4.1 of Dai and Meyn (1995)).

We let $A(t)$ and $D(t)$ denote the number of arrivals and departures on $[0,t]$, respectively. If an arrival occurs at time $t$, we let $U(t)$ be the interarrival time of the subsequent customer (so that the next arrival happens at $t + U(t)$). Similarly, if a departure occurs at time $t$, we let $S(t)$ be the service time of the next customer to be served.

For a counting process $\{N(t) : t \geq 0\}$ with event times $\{\tau_m\}_{m=1}^{\infty}$, we define

$$\int_0^t f(Z(t))dN(t) = \sum_{m=1}^{\infty} f(Z(\tau_m))1(\tau_m \leq t),$$

and we let $\Delta f(Z(t-)) = f(Z(t)) - f(Z(t-))$. Our first result is a stationary equation, or basic adjoint relationship (BAR), for the queue-length process.

LEMMA 1. *Initialize $Z(0) \sim Z$. If $\mathbb{E}|f(Z)| < \infty$ and, $Z(0)$-almost surely, $df(Z(t))/dt$ exists for almost all $t$, then, provided that all expectations are well defined,*

$$-\mathbb{E}(\partial_{r_a} f(Z)) - \mathbb{E}(1(X>0)\partial_{r_s} f(Z)) \\ + \mathbb{E}\int_0^1 \Delta f(Z(t-))dA(t) + \mathbb{E}\int_0^1 \Delta f(Z(t-))dD(t) = 0, \tag{1}$$

*where, for $f(z) = f(x, r_a, r_s)$, we define $\partial_{r_a} f(z) = df(z)/dr_a$ and $\partial_{r_s} f(z) = df(z)/dr_s$.*

*Proof of Lemma 1* For any function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions of the lemma, the fundamental theorem of calculus yields that $Z(0)$-almost surely,

$$f(Z(t)) - f(Z(0)) = -\int_0^t \big(\partial_{r_a} f(Z(t)) + 1(X(t)>0)\partial_{r_s} f(Z(t))\big)dt \\ + \int_0^t \Delta f(Z(t-))dA(t) + \int_0^t \Delta f(Z(t-))dD(t).$$

Setting $t = 1$ and taking expectations yields (1). $\square$

The following facts about the $G/G/1$ system will be useful. Though many of them are well known, we prove them using only the BAR.

LEMMA 2. *For any $m > 1$,*

$$\mathbb{E}A(1) = \mathbb{E}D(1) = \lambda, \quad \mathbb{P}(X>0) = \rho, \tag{2}$$

$$\mathbb{E}R_a^{m-1} = \lambda \mathbb{E}U^m/m, \quad \mathbb{E}(R_s^{m-1} 1(X>0)) = \lambda \mathbb{E}S^m/m, \quad \mathbb{E}(R_s^m | X=0) = \mathbb{E}S^m, \tag{3}$$

$$\mathbb{E}S\mathbb{E}\int_0^1 R_a(t)dD(t) + \mathbb{E}U\mathbb{E}\int_0^1 R_s(t)dA(t) \leq \mathbb{E}R_s + \mathbb{E}R_a \tag{4}$$

$$\mathbb{E}\int_0^1 1(X(t)=0)R_a(t)dD(t) = 1-\rho. \tag{5}$$

*Proof of Lemma 2* Let $f(z) = f(x, r_a, r_s) = r_a \wedge M$ and note that $\partial_{r_a} f(z) = 1(r_a < M)$ is defined everywhere except when $r_a = M$. To ensure that $\mathbb{E}\partial_{r_a} f(Z)$ is well defined, we choose $M > 0$ such that $R_a$ does not have a point mass at $M$ (any random variable can have at most countably many point masses). Using the BAR (7) yields

$$\begin{aligned}\mathbb{P}(R_a < M) &= \mathbb{E}\int_0^1 \big((R_a(t)\wedge M) - (R_a(t-)\wedge M)\big)dA(t) \\ &= \mathbb{E}\int_0^1 (U(t)\wedge M)dA(t) = \mathbb{E}(U\wedge M)\mathbb{E}A(1),\end{aligned}$$

where the last equality follows from the independence of $U(t)$ of the fact that any arrival happens at time $t$. Taking $M\to\infty$ yields $\mathbb{E}A(1) = \lambda$. Repeating this procedure with test functions $f(z) = x\wedge M$ and $f(z) = r_s\wedge M$ yields $\mathbb{E}D(1) = \mathbb{E}A(1)$ and $\mathbb{E}D(1) = \mu\mathbb{P}(X > 0)$, which proves (2). Using $f(z) = r_a^m \wedge M$ yields the first equality in (3) and using $f(z) = r_s^m\wedge M$ yields $m\mathbb{E}(R_s^{m-1}1(X > 0)) = \mathbb{E}S^m\mathbb{E}D(1) = \lambda\mathbb{E}S^m$. The third equality in (3) follows once we note that $\mathbb{E}(R_s^m 1(X = 0)) = \mathbb{E}S^m\mathbb{P}(X = 0)$ because when $X(t) = 0$, then $R_s(t)$ equals the service time of the next customer to arrive. Lastly, (4) and (5) follow from using the BAR with $f(z) = (r_a\wedge M)(r_s\wedge M)$ and $f(z) = 1(x = 0)(r_a\wedge M)$, respectively. $\square$

Instead of working directly with the queue length $X(t)$, we consider the compensated queue length

$$\widetilde{X}(t) = X(t) - \delta\lambda R_a(t) + \delta\mu R_s(t), \tag{6}$$

and let $\widetilde{X}$ be the random variable having its stationary distribution. The compensated queue length has the property that if an event (arrival or departure) occurs at time $t$, then $\Delta\widetilde{X}(t-)$ is independent of the fact that a jump happened at $t$, and $\mathbb{E}(\Delta\widetilde{X}(t-)) = 0$. For example, if an arrival happens at time $t$, then

$$\Delta\widetilde{X}(t-) = (X(t-) + \delta - \delta\lambda U(t) + \delta\mu R_s(t)) - (X(t-) + \delta\mu R_s(t-)) = \delta(1 - \lambda U(t)),$$

which equals zero in expectation. We comment on why working with the compensated queue length is necessary (or, at least, very helpful) at the end of Section 2.2. Specializing the BAR to functions of $\widetilde{X}$ yields

$$\begin{aligned}&\delta\mathbb{E}\big((\lambda-\mu 1(X>0))f'(X-\delta\lambda R_a+\delta\mu R_s)\big)\\&+\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dA(t)+\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dD(t)=0,\end{aligned} \tag{7}$$

provided that $\mathbb{E}|f(\widetilde{X})|<\infty$ and that all expectations are well defined.

The following key proposition converts (7) into an expression involving a second-order differential operator plus an error term; we prove it in Section 2.2. In the remainder of the paper, we write $f^{(k)}(\xi)$ to denote a generic $k$th-order Taylor expansion remainder term that may change from expression to expression.

PROPOSITION 1. *If $f\in C^3(\mathbb{R})$ with $\|f''\|<\infty$, then, provided that all expectations are well defined,*

$$\delta\mathbb{E}\big(\lambda-\mu 1(X>0)\big)f'(\widetilde{X})= -\mu\delta^2\mathbb{E}f'(X)+\mu\delta^2 f'(0)+\epsilon_0(f), \tag{8}$$

$$\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dA(t)=\frac{1}{2}\delta^2\lambda c_U^2\mathbb{E}f''(X)+\epsilon_A(f) \tag{9}$$

$$\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dD(t)=\frac{1}{2}\delta^2\mu c_S^2\mathbb{E}f''(X)+\epsilon_D(f), \tag{10}$$

*where*

$$\begin{aligned}\epsilon_0(f)=&\ -\mu\delta^3(-\lambda R_a+\mu R_s)f''(\xi)+\delta^2\mu\mathbb{E}\big(1(X=0)(-\lambda R_a+\mu R_s)f''(\xi)\big)\\ \epsilon_A(f)=&\ \frac{1}{6}\delta^3\mathbb{E}\int_0^1(1-\lambda U(t))^3 f'''(\xi(t))dA(t)+\frac{1}{2}\delta^3 c_U^2\mathbb{E}\int_0^1 \mu R_s(t)f'''(\xi(t-))dA(t)\\&-\frac{1}{2}\delta^2\lambda c_U^2\mathbb{E}\int_0^1\int_0^{U(t)}\big(X(t+u)-X(t-)\big)f'''(\xi(t+u))dudA(t),\end{aligned}$$

$$
\begin{aligned}
\epsilon_D(f) = & \frac{1}{6}\delta^3 \mathbb{E}\int_0^1 (1-\mu S(t))^3 f'''(\xi(t))dD(t) - \frac{1}{2}\delta^3 c_S^2 \mathbb{E}\int_0^1 \lambda R_a(t) f'''(\xi(t-))dD(t) \\
& - \frac{1}{2}\delta^3 \mu c_S^2 f''(\delta) \\
& - \frac{1}{2}\delta^2 \mu c_S^2 \mathbb{E}\int_0^1 \int_0^{1(X(t)=0)R_a(t)+S(t)} \big(X(t+u)-X(t-)\big) f'''(\xi(t+u))dudD(t).
\end{aligned}
$$

Combining the first- and second-order terms in (8)–(10) suggests a diffusion generator of the form

$$
G_Y f(x) = -\theta f'(x) + \frac{1}{2}\sigma^2 f''(x) + \theta f'(0), \quad x \geq 0,
$$

where $\theta = \mu\delta^2$ and $\sigma^2 = \delta^2(\lambda c_U^2 + \mu c_S^2)$. This generator corresponds to a one-dimensional reflected Brownian motion (RBM) on the nonnegative real numbers; the presence of $f'(0)$ is due to the reflection at zero Harrison and Reiman (1981). We let $Y$ be an exponentially distributed random variable with rate $2\theta/\sigma^2$, which corresponds to the stationary distribution of this RBM.

### 2.1. The Stein method step

Given $h : \mathbb{R} \to \mathbb{R}$ with $E|h(Y)| < \infty$, let $f_h(x)$ solve the Poisson equation

$$
\begin{aligned}
-\theta f_h'(x) + \frac{1}{2}\sigma^2 f_h''(x) &= \mathbb{E}h(Y) - h(x), \quad x \in \mathbb{R}, \\
f_h'(0) &= 0. 
\end{aligned} \tag{11}
$$

This is the Poisson equation for the exponential distribution and it is typically stated only for $x \geq 0$, but extending its solution to the entire real line is trivial. The condition that $f_h'(0) = 0$ is usually omitted, because it is automatically satisfied Ross (2011). The following is Lemma 1 of Braverman and Scully (2024).

LEMMA 3. *If $h \in \mathrm{Lip}(1)$, then $f_h'(0)=0$, $f_h'''(x)$ is absolutely continuous, and*

$$\|f_h''\| \leq 1/\theta \quad \text{and} \quad \|f_h'''\| \leq 4/\sigma^2.$$

*As a consequence, $|f_h'(x)| \leq |x|/\theta$ and $|f_h(x)| \leq \frac{1}{2}x^2/\theta$ for all $x \in \mathbb{R}$.*

The following result contains the diffusion approximation error bound.

THEOREM 1. *For any $h \in \mathrm{Lip}(1)$,*

$$|\mathbb{E}h(X) - \mathbb{E}h(Y)| \leq |\epsilon_0(f_h)| + |\epsilon_A(f_h)| + |\epsilon_D(f_h)|. \tag{12}$$

*Furthermore,*

$$\begin{aligned}
|\epsilon_0(f_h)| &\leq \delta\big(\lambda^2\mathbb{E}U^2/2 + \big(\delta + \mu\lambda\mathbb{E}S^2/2\big) + \lambda\mathbb{E}(R_a|X=0) + 1\big), \\
|\epsilon_A(f_h)| &\leq 2\delta^3\sigma^{-2}\lambda\Big(\frac{1}{3}\mathbb{E}\,|1-\lambda U|^3 + c_U^2\big(\mu\lambda\mathbb{E}U^2/2 + \delta + \mu\lambda\mathbb{E}S^2/2\big) \\
&\qquad + c_U^2(2 + \mu\lambda\mathbb{E}U^2/2 + \mu^2\mathbb{E}S^2)\Big) \\
|\epsilon_D(f_h)| &\leq 2\delta^3\sigma^{-2}\Big(\mathbb{E}\,|1-\mu S|^3\,\lambda + c_S^2\lambda(\mu\lambda\mathbb{E}U^2/2 + \delta + \mu\lambda\mathbb{E}S^2/2) \\
&\qquad + c_S^2\mu\big(\delta^2 + \delta(2 + \lambda^2\mathbb{E}S^2/2 + \lambda^2\mathbb{E}U^2)\big)\Big) + \frac{1}{2}\delta c_S^2.
\end{aligned}$$

One can argue that $\mathbb{E}(R_a|X=0)$, which appears in the bound of $|\epsilon_0(f_h)|$, equals $\mathbb{E}I^2/(2\mathbb{E}I)$, where $I$ is the length of the idle period following the busy period initiated by an arrival to an empty system. Bounding $\mathbb{E}I^2/(2\mathbb{E}I)$ in terms of the $G/G/1$ primitives is known to be a difficult problem; e.g., Li and Ou (1995), Wolff and Wang (2003), Blanchet and Glynn (2006). Some progress was made by Braverman and Scully (2024), which derives a bound in terms of the first three moments of $U$ and $S$ and the hazard rate of $U$ for some classes of interarrival time distributions.

For a cruder bound, we can use a trick inspired by Dai et al. (2025). Note that $\delta = (1-\rho) = \mathbb{P}(X=0)$ and that

$$\delta \mathbb{E}(R_a | X=0) = \mathbb{E}(R_a 1(X=0)) \leq \sqrt{\mathbb{P}(X=0)}\sqrt{\mathbb{E}(R_a^2 1(X=0))} \leq \delta^{1/2}\lambda \mathbb{E}U^3/3,$$

where the final inequality follows from Lemma 2. The $\delta^{1/2}$ term can be improved to $\delta^{1-\epsilon}$ for arbitrarily small $\epsilon > 0$ using the

$$\mathbb{E}(R_a^2 1(X=0)) \leq \sqrt{\mathbb{P}(X=0)}\sqrt{\mathbb{E}(R_a^4 1(X=0))} \leq \dots$$

provided that enough moments of $U$ exist.

*Proof of Theorem 1* Fix $h \in \mathrm{Lip}(1)$ and let $f_h(x)$ be the solution to (11). We verify that the BAR (7) holds with $f(x) = f_h(x)$, because combining it with Proposition 1 implies (12). Since $\mathbb{E}R_a^2, \mathbb{E}R_s^2 < \infty$ (by Lemma 2 and the assumption that $\mathbb{E}U^3, \mathbb{E}S^3 < \infty$) and $\mathbb{E}X^2 < \infty$, the bound on $|f_h(x)|$ in Lemma 3 yields $\mathbb{E}|f_h(\widetilde{X})| < \infty$; the fact that $\mathbb{E}|f_h'(\widetilde{X})| < \infty$ is argued similarly. To show that the integral terms in (7) are well defined, e.g., $\mathbb{E}\int_0^1 |\Delta f_h(\widetilde{X}(t-))| dA(t) < \infty$, one can use the fact that $|f_h'(x)| \leq |x|/\theta$ together with (4) of Lemma 2 and the bound $X(t) \leq X(0) + \delta A(1)$ (for $t \leq 1$).

The bound on $|\epsilon_0(f_h)|$ follows from the following facts: $f_h'(0) = 0$ and $\|f_h''\| \leq \mu^{-1}\delta^{-2}$ (Lemma 3), $\mathbb{P}(X=0) = 1-\rho$, $\mathbb{E}R_a = \lambda \mathbb{E}U^2/2$, $\mathbb{E}R_s = (1-\rho)\mathbb{E}S + \lambda \mathbb{E}S^2/2$, and $\mathbb{E}(R_s | X = 0) = \mathbb{E}S$ (Lemma 2).

The bound on $\epsilon_A(f_h)$ requires additional ingredients: $\|f_h'''\| \leq 4\sigma^{-2}$ (Lemma 3), $\mathbb{E}A(1) = \lambda$ and $\mathbb{E}\int_0^1 \mu R_s(t) dA(t) \leq \lambda\mu \mathbb{E}(R_a + R_s)$ (Lemma 2), and the fact that

$$\mathbb{E}\int_0^1 \int_0^{U(t)} |X(t+u) - X(t-)| \, du dA(t) \leq \mathbb{E}\int_0^1 \int_0^{U(t)} \delta\big(1 + (D(t+u) - D(t-))\big) du dA(t)$$

$$
\begin{aligned}
&\le \mathbb{E}\int_0^1 \int_0^{U(t)} \delta\big(1+1+\mathbb{E}S(t,t+u]\big)dudA(t)\\
&\le \delta\mathbb{E}\int_0^1 \int_0^{U(t)} \big(1+1+\mu u+\mu^2\mathbb{E}S^2\big)dudA(t)\\
&= \delta\mathbb{E}\int_0^1 \big(2\mathbb{E}U+\mu\mathbb{E}U^2/2+\mu^2\mathbb{E}S^2\mathbb{E}U\big)dA(t)\\
&= \delta(2+\mu\lambda\mathbb{E}U^2/2+\mu^2\mathbb{E}S^2),
\end{aligned}
$$

where in the second inequality, $S(t,t+u]$ is the number of potential service completions on $(t,t+u]$ provided that service was completed exactly at time $t$, which is independent of all other quantities in the integral, and the third inequality is Lorden's inequality (Lorden (1970)) applied to $\mathbb{E}S(t,t+u]$. The bound on $\epsilon_D(f_h)$ uses $\mathbb{E}D(1)=\lambda$ and $\mathbb{E}\int_0^1 \mu R_a(t)dD(t) \le \lambda\mu\mathbb{E}(R_a+R_s)$ (Lemma 2) and the fact that

$$
\begin{aligned}
&\mathbb{E}\int_0^1 \int_0^{1(X(t)=0)R_a(t)+S(t)} |X(t+u)-X(t-)|\,dudD(t)\\
&= \mathbb{E}\int_0^1 \delta 1(X(t)=0)R_a(t)dD(t) + \mathbb{E}\int_0^1 \int_{1(X(t)=0)R_a(t)}^{1(X(t)=0)R_a(t)+S(t)} |X(t+u)-X(t-)|\,dudD(t)\\
&\le \delta^2 + \mathbb{E}\int_0^1 \delta(2\mathbb{E}S+\lambda\mathbb{E}S^2/2+\lambda^2\mathbb{E}U^2\mathbb{E}S)dD(t)\\
&= \delta^2+\delta(2+\lambda^2\mathbb{E}S^2/2+\lambda^2\mathbb{E}U^2),
\end{aligned}
$$

where the inequality is due to (5) and Lorden's inequality, and in the last equality we used $\mathbb{E}D(1)=\lambda$ and $\lambda\mathbb{E}S=\rho<1$. □

### 2.2. Extracting the diffusion generator

A critical element in the proof of Proposition 1 is the following relationship between event-average expectations and time-average expectations. It is a special case of the Palm inversion formula (Baccelli and Brémaud 2003, Equation (1.2.25)).

LEMMA 4. *For any bounded $f:\mathbb{R}\to\mathbb{R}$,*

$$\mathbb{E}f(X)=\mathbb{E}\int_0^1\int_0^{U(t)} f(X(t+u))dudA(t) \qquad \textit{and} \tag{13}$$

$$\mathbb{E}f(X)=\mathbb{E}\int_0^1\int_0^{1(X(t)=0)R_a(t)+S(t)} f(X(t+u))dudD(t). \tag{14}$$

*Proof of Lemma 4* Fix a bounded function $f:\mathbb{R}\to\mathbb{R}$, initialize $Z(0)$ according to its stationary distribution, and note that

$$\mathbb{E}f(X)=\mathbb{E}\int_0^1 f(X(t))dt.$$

We first prove (13). Let $\tau_m$ be the time of the $m$th arrival and observe that

$$\int_0^1 f(X(t))dt=\int_0^{\tau_1} f(X(t))dt+\sum_{m=1}^{A(1)}\int_{\tau_m}^{\tau_{m+1}} f(X(t))dt-\int_1^{\tau_{A(1)+1}} f(X(t))dt.$$

Since $\tau_{m+1}-\tau_m=U(\tau_m)$, it follows that

$$\begin{aligned}\sum_{m=1}^{A(1)}\int_{\tau_m}^{\tau_{m+1}} f(X(t))dt &= \sum_{m=1}^{A(1)}\int_0^{U(\tau_m)} f\big(X(\tau_m+u)\big)du \\ &= \int_0^1\int_0^{U(t)} f\big(X(t+u)\big)dudA(t).\end{aligned}$$

Furthermore, since $\tau_1=R_a(0)$ and $\tau_{A(1)+1}=1+R_a(1)$,

$$\int_0^{\tau_1} f(X(t))dt-\int_1^{\tau_{A(1)+1}} f(X(t))dt=\int_0^{R_a(0)} f(X(t))dt-\int_0^{R_a(1)} f(X(1+t))dt,$$

and (13) follows once we note that the expected value of this expression is zero by stationarity and the fact that $\mathbb{E}R_a<\infty$ by Lemma 2.

Let $R_d(t)$ be the time until the first departure after time $t$ and let $R_d$ have its stationary distribution. The proof of (14) is identical to (13) once we observe that $\mathbb{E}R_d<\infty$, because $R_d(t)=1(X(t)=0)R_a(t)+R_s(t)$ and, consequently, $\mathbb{E}R_d\le\mathbb{E}R_a+\mathbb{E}R_s<\infty$ by Lemma 2.

$\square$

After proving Proposition 1, we comment on why it is essential to work with the compensated $\widetilde{X}(t)$ instead of $X(t)$.

*Proof of Proposition 1* We first prove (8), which follows from some basic algebra:

$$\begin{aligned}
&\delta\mathbb{E}\big(\lambda-\mu 1(X>0)\big)f'(X-\delta\lambda R_a+\delta\mu R_s)\\
=&\ \delta(\lambda-\mu)\mathbb{E}f'(X-\delta\lambda R_a+\delta\mu R_s)+\delta\mu\mathbb{E}\big(1(X=0)f'(-\delta\lambda R_a+\delta\mu R_s)\big)\\
=&\ -\mu\delta^2\mathbb{E}\big(f'(X)+\delta(-\lambda R_a+\mu R_s)f''(\xi)\big)+\delta\mu\mathbb{E}\big(1(X=0)\big(f'(0)-\delta(\lambda R_a-\mu R_s)f''(\xi)\big)\\
=&\ -\mu\delta^2\mathbb{E}f'(X)+\mu\delta^2 f'(0)+\epsilon_0(f),
\end{aligned}$$

where in the last equality we used $\mathbb{P}(X=0)=1-\rho$ (Lemma 2). Next, we prove (9). If an arrival happens at time $t$, then $\Delta\widetilde{X}(t-)$ is independent of the history of $Z(t)$ on $[0,t)$, and $\Delta\widetilde{X}(t-)=\delta(1-\lambda U(t))\stackrel{d}{=}\delta(1-\lambda U)$. Since $\mathbb{E}(1-\lambda U)=0$,

$$\begin{aligned}
\mathbb{E}\int_0^1\Delta f(\widetilde{X}(t-))dA(t)=&\ \frac{1}{2}\delta^2\mathbb{E}(1-\lambda U)^2\mathbb{E}\int_0^1 f''(\widetilde{X}(t-))dA(t)\\
&+\frac{1}{6}\delta^3\mathbb{E}\int_0^1(1-\lambda U(t))^3 f'''(\xi(t))dA(t),
\end{aligned}\tag{15}$$

and the first term expands into

$$\begin{aligned}
&\frac{1}{2}\delta^2\mathbb{E}(1-\lambda U)^2\mathbb{E}\int_0^1 f''(X(t-))dA(t)\\
&+\frac{1}{2}\delta^3\mathbb{E}(1-\lambda U)^2\mathbb{E}\int_0^1\big(\widetilde{X}(t-)-X(t-)\big)f'''(\xi(t-))dA(t)
\end{aligned}$$

Noting that $\mathbb{E}(1-\lambda U)^2=c_U^2$ and $\widetilde{X}(t-)-X(t-)=\mu R_s(t)$, to conclude (9) we now show that

$$\begin{aligned}
&\mathbb{E}\int_0^1 f''(X(t-))dA(t)\\
=&\ \lambda\mathbb{E}f''(X)-\lambda\mathbb{E}\int_0^1\int_0^{U(t)}\big(X(t+u)-X(t-)\big)f'''(\xi(t+u))dudA(t),
\end{aligned}$$

which follows from expanding the right-hand side of (13) of Lemma 4:

$$\begin{aligned}\mathbb{E}f''(X) = &\ \mathbb{E}\int_0^1 U(t)f''(X(t-))dA(t)\\ &+\mathbb{E}\int_0^1\int_0^{U(t)} \big(X(t+u)-X(t-)\big)f'''(\xi(t+u))dudA(t).\end{aligned}$$

Let us now prove (10). Proceeding as with (9), we first have

$$\begin{aligned}\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dD(t) = &\ \frac{1}{2}\delta^2\mathbb{E}(1-\mu S)^2\mathbb{E}\int_0^1 f''(X(t-))dD(t)\\ &+\frac{1}{2}\delta^3\mathbb{E}(1-\mu S)^2\mathbb{E}\int_0^1 \big(\widetilde{X}(t-)-X(t-)\big)f'''(\xi(t-))dD(t)\\ &+\frac{1}{6}\delta^3\mathbb{E}\int_0^1 (1-\mu S(t))^3 f'''(\xi(t))dD(t).\end{aligned}$$

Since $\mathbb{E}(1-\mu S)^2 = c_S^2$ and $\widetilde{X}(t-)-X(t-) = -\lambda R_a(t)$, it suffices to show that

$$\begin{aligned}&\mathbb{E}\int_0^1 f''(X(t-))dD(t)\\ &= \mu\mathbb{E}f''(X) - \mu f''(\delta)(1-\rho)\\ &\quad -\mathbb{E}\int_0^1\int_0^{1(X(t)=0)R_a(t)+S(t)} \big(X(t+u)-X(t-)\big)f'''(\xi(t+u))dudD(t).\end{aligned}$$

The latter equality follows from first using Lemma 4 to get

$$\begin{aligned}\mathbb{E}f''(X) = &\ \mathbb{E}\int_0^1 (1(X(t)=0)R_a(t)+S(t))f''(X(t-))dD(t)\\ &+\mathbb{E}\int_0^1\int_0^{1(X(t)=0)R_a(t)+S(t)} \big(X(t+u)-X(t-)\big)f'''(\xi(t+u))dudD(t),\end{aligned}$$

and then noting that the first term equals

$$\begin{aligned}&f''(\delta)\mathbb{E}\int_0^1 1(X(t)=0)R_a(t)dD(t)+\mathbb{E}\int_0^1 S(t)f''(X(t-))dD(t)\\ &= f''(\delta)(1-\rho)+\mathbb{E}S\mathbb{E}\int_0^1 f''(X(t-))dD(t),\end{aligned}$$

where the equality comes from (5) of Lemma 2. □

Note that the expansion of $\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dA(t)$ in (15) does not contain a term involving $f'(\widetilde{X}(t-))$. Had we started with the BAR for $X$ instead of the compensated $\widetilde{X}$, then the expansion of $\mathbb{E}\int_0^1 \Delta f(X(t-))dA(t)$ would have included a term of the form $\delta\mathbb{E}\int_0^1 f'(X(t-))dA(t)$. Attempting to relate the latter quantity to $\mathbb{E}f'(X)$ using Lemma 4 results in

$$\begin{aligned}\mathbb{E}f'(X) = {} & \mathbb{E}\int_0^1 U(t)f'(X(t-))dA(t)\\ & + \mathbb{E}\int_0^1 f''(X(t-))\int_0^{U(t)} \big(X(t+u) - X(t-)\big)dudA(t)\\ & + \frac{1}{2}\mathbb{E}\int_0^1 \int_0^{U(t)} \big(X(t+u) - X(t-)\big)^2 f'''(\xi(t+u))dudA(t).\end{aligned}$$

While the third term is an error term that can be bounded, the second term needs to be evaluated and included in the approximating diffusion generator. However, this is a difficult task, because $X(t+u) - X(t-)$ depends on the number of departures on $[t, t+u]$, which could itself depend, in a nontrivial way, on the fact that an arrival happens at $t$. A similar problem occurs in the expansion of $\mathbb{E}\int_0^1 \Delta f(X(t-))dD(t)$. This challenge was not present in Braverman and Scully (2024) where it was simple to compute $X(t+u) - X(t-)$ (although Braverman and Scully (2024) still worked with the compensated workload out of algebraic convenience).

## 3. The join-the-shortest-queue system

Consider a parallel-server system with $n$ identical servers, each with their own buffer, operating under a first-come-first-served policy. Customers arrive according to a renewal process and customer service times are i.i.d. Let $U$ and $S$ be random variables having the interarrival and service time distributions, respectively. Upon arrival, customers are routed to the server with the shortest queue; ties are broken uniformly at random. We assume that

$$\mathbb{E}U^3 < \infty \quad \text{and} \quad \mathbb{E}S^3 < \infty,$$

and, for simplicity, that simultaneous events do not occur with probability one. Set

$$\lambda = 1/\mathbb{E}U, \quad \mu = 1/\mathbb{E}S, \quad \rho = \lambda/n\mu,$$
$$c_U^2 = \lambda^2 \mathrm{Var}(U), \quad c_S^2 = \mu^2 \mathrm{Var}(S),$$

and assume that $\rho < 1$.

Let $Q_i(t)$ be the number of customers (both in service and waiting) at server $i$ at time $t \geq 0$. Also let $R_a(t)$ be the time until the next arrival and let $R_{s,i}(t)$ be the remaining service time of the customer being served by server $i$; when server $i$ is idle, $R_{s,i}(t)$ denotes the service time of the next customer to arrive. Define

$$Q(t) = (Q_1(t), \dots, Q_n(t)), \quad R_s(t) = (R_{s,1}(t), \dots, R_{s,n}(t)), \quad \text{and}$$
$$Z(t) = (Q(t), R_a(t), R_s(t)),$$

and let $X(t) = \delta \sum_{i=1}^n Q_i(t)$ be the scaled total customer count, where $\delta = (1-\rho)$. We are interested in approximating the total number of customers in the system as $\rho \to 1$. We assume that $\{Z(t) : t \geq 0\}$ is positive Harris recurrent (see Bramson (2011) for sufficient conditions) and let $Z = (Q, R_a, R_s)$ be the vector having its stationary distribution.

Let $A(t)$ denote the number of arrivals on $[0,t]$, and let $D_i(t)$ denote the number of departures from server $i$ on $[0,t]$. If an arrival occurs at time $t$, we let $U(t)$ be the interarrival time of the subsequent customer. Similarly, if a departure occurs from server $i$ at time $t$, we let $S_i(t)$ be the service time of the next customer to be served. We also let $\Lambda_i(t)$ denote the (random) remaining time until a customer gets routed to server $i$. The following BAR for $Z$ is established identically to Lemma 1.

Lemma 5. *Initialize $Z(0) \sim Z$. If $\mathbb{E}|f(Z)| < \infty$ and, $Z(0)$-almost surely, $df(Z(t))/dt$ exists for almost all $t$, then, provided that all expectations are well defined,*

$$\begin{aligned}
&-\mathbb{E}(\partial_{r_a} f(Z)) + \sum_{i=1}^{n} \mathbb{E}(1(Q_i > 0)\partial_{r_{s,i}} f(Z)) \\
&+\mathbb{E}\int_0^1 \Delta f(Z(t-))dA(t) + \sum_{i=1}^{n} \mathbb{E}\int_0^1 \Delta f(Z(t-))dD_i(t) = 0,
\end{aligned} \tag{16}$$

*where, for $f(z) = f(q_1, \ldots, q_n, r_a, r_{s,1}, \ldots, r_{s,n})$, we define $\partial_{r_a} f(z) = df(z)/dr_a$ and $\partial_{r_{s,i}} f(z) = df(z)/dr_{s,i}$*

The following facts about the JSQ system will be useful.

Lemma 6. *For any $1 \le i \le n$ and $m > 1$,*

$$\mathbb{E}A(1) = n\mathbb{E}D(1) = \lambda, \quad \mathbb{P}(Q_i > 0) = \rho, \tag{17}$$

$$\mathbb{E}R_a^{m-1} = \lambda \mathbb{E}U^m/m, \quad \mathbb{E}(R_{s,i}^{m-1}1(Q_i > 0)) = (\lambda/n)\mathbb{E}S^m/m, \quad \mathbb{E}(R_{s,i}^m | Q_i = 0) = \mathbb{E}S^m. \tag{18}$$

*Proof of Lemma 6* Proceeding as in the proof of Lemma 2, using $f(z) = r_a \wedge M$, $f(z) = (q_1 + \cdots + q_n) \wedge M$ and $f(z) = r_s \wedge M$ yields $\mathbb{E}A(1) = \lambda$, $n\mathbb{E}D(1) = \mathbb{E}A(1)$, and $\mathbb{E}S\mathbb{E}D(1) = \mathbb{P}(X > 0)$, respectively, which proves (17). Using $f(z) = r_a^m \wedge M$ yields the first equality in (18) and using $f(z) = r_{s,i}^m \wedge M$ yields $m\mathbb{E}(R_{s,i}^{m-1}1(Q_i > 0)) = \mathbb{E}S^m\mathbb{E}D(1) = (\lambda/n)\mathbb{E}S^m$. The third equality in (3) follows once we note that $\mathbb{E}(R_{s,i}^m 1(Q_i = 0)) = \mathbb{E}S^m\mathbb{P}(Q_i = 0)$ because when $Q_i(t) = 0$, then $R_{s,i}(t)$ equals the service time of the next customer to arrive. $\square$

Define the compensated total customer count

$$\widetilde{X}(t) = X(t) - \delta\lambda R_a(t) + \sum_{i=1}^{n} \delta\mu R_{s,i}(t).$$

Specialized to $\widetilde{X}$, the BAR (16) becomes

$$\begin{aligned}&\delta\mathbb{E}\Big(\Big(\lambda - \sum_{i=1}^{n} 1(Q_i > 0)\mu\Big) f'(\widetilde{X})\Big) + \mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dA(t)\\&+\sum_{i=1}^{n}\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dD_i(t) = 0. \qquad (19)\end{aligned}$$

The following result extracts the diffusion generator from (19) analogously to Proposition 1.

Proposition 2. *If $f \in C^3(\mathbb{R})$ with $\|f''\| < \infty$, then, provided that all expectations are well defined,*

$$\delta\mathbb{E}\Big(\Big(\lambda - \sum_{i=1}^{n} 1(Q_i > 0)\mu\Big) f'(\widetilde{X})\Big) = -n\mu\delta^2\mathbb{E}f'(X) + n\mu\delta^2 f'(0) + \epsilon_0(f), \qquad (20)$$

$$\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dA(t) = \frac{1}{2}\delta^2\lambda c_U^2\mathbb{E}f''(X) + \epsilon_A(f) \qquad (21)$$

$$\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dD_i(t) = \frac{1}{2}\delta^2\mu c_S^2\mathbb{E}f''(X) + \epsilon_{D,i}(f), \qquad (22)$$

*where*

$$\begin{aligned}\epsilon_0(f) =& -n\mu\delta^3\mathbb{E}\Big(\Big(-\lambda R_a + \sum_{i=1}^{n}\mu R_{s,i}\Big) f''(\xi)\Big)\\&+\delta^2\mu\sum_{i=1}^{n}\mathbb{E}\Big(1(Q_i = 0)\Big(\sum_{j=1}^{n} Q_j - \lambda R_a + \sum_{j=1}^{n}\mu R_{s,j}\Big) f''(\xi)\Big)\\ \epsilon_A(f) =& \frac{1}{6}\delta^3\mathbb{E}\int_0^1 (1-\lambda U(t))^3 f'''(\xi(t))dA(t)\\&+\frac{1}{2}\delta^3 c_U^2\mathbb{E}\int_0^1 \Big(\sum_{i=1}^{n}\mu R_{s,i}(t)\Big) f'''(\xi(t-))dA(t)\\&-\frac{1}{2}\delta^2\lambda c_U^2\mathbb{E}\int_0^1\int_0^{U(t)} \big(X(t+u) - X(t-)\big) f'''(\xi(t+u))dudA(t),\end{aligned}$$

*and*

$$\begin{aligned}\epsilon_{D,i}(f) &= \frac{1}{6}\delta^3\mathbb{E}\int_0^1 (1-\mu S(t))^3 f'''(\xi(t))dD_i(t) \\ &+ \frac{1}{2}\delta^3 c_S^2\mathbb{E}\int_0^1 \Big(-\lambda R_a(t) + \sum_{j\neq i}\mu R_{s,j}(t)\Big) f'''(\xi(t-))dD_i(t) \\ &- \frac{1}{2}\delta^2\mu c_S^2\mathbb{E}\int_0^1 1(Q_i(t)=0)\Lambda_i(t) f''(X(t-))dD_i(t) \\ &- \frac{1}{2}\delta^2\mu c_S^2\mathbb{E}\int_0^1\int_0^{1(Q_i(t)=0)\Lambda_i(t)+S_i(t)} \big(X(t+u)-X(t-)\big) f'''(\xi(t+u))du dD_i(t).\end{aligned}$$

Proposition 2 suggests a diffusion approximation with generator

$$G_Y f(x) = -\theta f'(x) + \frac{1}{2}\sigma^2 f''(x) + \theta f'(0), \quad x\geq 0,$$

where $\theta = n\mu\delta^2$ and $\sigma^2 = \delta^2(\lambda c_U^2 + n\mu c_S^2)$, which, like in Section 2, corresponds to the exponential distribution with rate $2\theta/\sigma^2$. We do not bound the diffusion approximation error like we did in Theorem 1. Instead, we comment on the error terms that are new to Proposition 2, compared to Proposition 1.

There are two novel terms. First, Proposition 2 has a term of the form $\|f''\|\mathbb{E}\big(1(Q_i = 0)\sum_{j=1}^n Q_j\big)$ appearing in $\epsilon_0(f)$. To obtain a rate of convergence similar to Theorem 1, i.e., show that the error decays at a rate of $\delta$, one needs to show that $\mathbb{E}\big(1(Q_i=0)\sum_{j=1}^n Q_j\big)$ is of order $\delta$. This is a state-space collapse result, because one needs to show that $Q_j$ are all close to $Q_i$, which equals zero. The work by Dai et al. (2024) would be a good starting point.

Second, some additional effort is required to bound

$$\mathbb{E}\int_0^1 1(Q_i(t)=0)\Lambda_i(t) f''(X(t-))dD_i(t),$$

which appears in the third line of the expression for $\epsilon_{D,i}(f)$. One can show using Lemma 7 that

$$\begin{aligned}&\mathbb{E}\int_0^1 1(Q_i(t)=0)\Lambda_i(t)f''(X(t-))dD_i(t)\\ &=\mathbb{E}\big(1(Q_i=0)f''(X)\big)\\ &\quad-\mathbb{E}\int_0^1\int_0^{1(Q_i(t)=0)\Lambda_i(t)} 1(Q_i(t)=0)(X(t+u)-X(t-))f'''(\xi(t+u))dudD_i(t),\end{aligned}$$

and the right-hand side can be bounded exploiting the fact that $\mathbb{P}(Q_i=0)=1-\rho$ ( Lemma 6) and a bound on $|X(t+u)-X(t-)|$ similar to the one used in the proof of Theorem 1. We prove Proposition 2 in Section 3.1, but first we present an analog of Lemma 4.

Lemma 7. *For any bounded $f:\mathbb{R}\to\mathbb{R}$,*

$$\mathbb{E}f(X)=\mathbb{E}\Big(\int_0^1\int_0^{U(t)} f(X(t+u))dudA(t)\Big)\quad \textit{and} \tag{23}$$

$$\mathbb{E}f(X)=\mathbb{E}\Big(\int_0^1\int_0^{1(Q_i(t)=0)\Lambda_i(t)+S_i(t)} f(X(t+u))dudD_i(t)\Big). \tag{24}$$

*Proof of Lemma 7* Repeating the proof of Lemma 4 and noting that $\mathbb{E}R_a<\infty$ (Lemma 6) yields (23). Let $R_{d,i}(t)=1(Q_i(t)=0)\Lambda_i(t)+R_{s,i}(t)$ be the remaining time until the next departure from server $i$. Since $\mathbb{E}R_a<\infty$, our equiprobable routing tie-breaker rule implies that $\mathbb{E}(1(Q_i=0)\Lambda_i)<\infty$ (where $\Lambda_i$ has the steady-state distribution of $\{\Lambda_i(t):t\geq 0\}$). Combining this with the fact that $\mathbb{E}R_{s,i}<\infty$ (Lemma 6) yields the finiteness of the steady-state expected departure time. □

### 3.1. Extracting diffusion generator

*Proof of Proposition 2* We first prove (20):

$$\delta\mathbb{E}\Big((\lambda-\sum_{i=1}^n 1(Q_i>0)\mu)f'(\widetilde{X})\Big)=-n\mu\delta(1-\rho)\mathbb{E}f'(\widetilde{X})+\delta\mu\sum_{i=1}^n\mathbb{E}\big(1(Q_i=0)f'(\widetilde{X})\big).$$

Recalling that $\delta = (1-\rho)$ and $\widetilde{X} = \delta\sum_{i=1}^n Q_i - \delta\lambda R_a + \delta\sum_{i=1}^n \mu R_{s,i}$, the first term on the right-hand side equals

$$-n\mu\delta^2\mathbb{E}f'(\widetilde{X}) = -n\mu\delta^2\mathbb{E}f'(X) - n\mu\delta^3\mathbb{E}\Big(\Big(-\lambda R_a + \sum_{i=1}^n \mu R_{s,i}\Big)f''(\xi)\Big),$$

and the second term equals

$$\begin{aligned}
&\delta\mu\sum_{i=1}^n \mathbb{E}\big(1(Q_i=0)f'(\widetilde{X})\big)\\
&= \delta\mu\sum_{i=1}^n \mathbb{E}\Big(1(Q_i=0)\big(f'(0) + \delta\Big(\sum_{i=1}^n Q_i - \lambda R_a + \sum_{i=1}^n \mu R_{s,i}\Big)f''(\xi)\big)\Big),
\end{aligned}$$

which proves (20). The proof of (21) is identical to the proof of (9) in Proposition 1. The proof of (22) is similar to that of (10). Namely,

$$\begin{aligned}
\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dD_i(t) = {}& \frac{1}{2}\delta^2\mathbb{E}(1-\mu S)^2\mathbb{E}\int_0^1 f''(X(t-))dD_i(t)\\
&+\frac{1}{2}\delta^3\mathbb{E}(1-\mu S)^2\mathbb{E}\int_0^1 \big(\widetilde{X}(t-) - X(t-)\big)f'''(\xi(t-))dD_i(t)\\
&+\frac{1}{6}\delta^3\mathbb{E}\int_0^1 (1-\mu S_i(t))^3 f'''(\xi(t))dD_i(t),
\end{aligned}$$

where $\widetilde{X}(t-) - X(t-) = -\delta\lambda R_a(t) + \sum_{j\neq i}\delta\mu R_{s,j}(t)$. We conclude by using Lemma 7 to get

$$\begin{aligned}
\mathbb{E}f''(X) = {}& \mathbb{E}\int_0^1 1(Q_i(t)=0)\Lambda_i(t)f''(X(t-))dD_i(t) + \mathbb{E}S\int_0^1 f''(X(t-))dD_i(t)\\
&+\mathbb{E}\int_0^1\int_0^{1(Q_i(t)=0)\Lambda_i(t)+S_i(t)} \big(X(t+u) - X(t-)\big)f'''(\xi(t+u))du dD_i(t).
\end{aligned}$$

$\square$

# 4. The tandem queue

Consider two single-server stations in tandem, operating under a first-come-first-served policy. Customers arrive to the first station according to a renewal process and, after being served, move onto the second station. After being served at the second station, customers leave the system. Service times at station $i$ are i.i.d. and independent of both the arrival process and service times at other stations. Let $U$ and $S_i$ be random variables corresponding to the interrival time and service time at station $i$, respectively. We assume that

$$\mathbb{E}U^3 < \infty \quad \text{and} \quad \mathbb{E}S_1^3, \mathbb{E}S_2^3 < \infty,$$

and, for simplicity, that simultaneous events do not occur with probability one. Set

$$\lambda = 1/\mathbb{E}U, \quad \mu_i = 1/\mathbb{E}S_i, \quad \rho_i = \lambda/\mu_i,$$
$$c_U^2 = \lambda^2 \mathrm{Var}(U), \quad c_{S,i}^2 = \mu_i^2 \mathrm{Var}(S_i),$$

and assume that $\rho_i < 1$ for $i = 1, 2$.

At time $t \geq 0$, let $Q_i(t)$ denote the queue-length at station $i$, let $R_a(t)$ denote the residual interarrival time, let $R_{s,i}(t)$ denote the remaining service time of the customer in service at station $i$ and, if $Q_i(t) = 0$, let it denote the service time of the next customer to enter service. Set $X(t) = (\delta_1 Q_1(t), \delta_2 Q_2(t))$, where $\delta_i = 1 - \rho_i$, and $R_s(t) = (R_{s,1}(t), R_{s,2}(t))$, and define the queue-length process

$$\{Z(t) = (X(t), R_a(t), R_s(t)) : t \geq 0\}.$$

We assume that this process is positive Harris recurrent (sufficient conditions are discussed in Dai and Meyn (1995)) and let $Z = (X, R_a, R_s)$ be the vector having the stationary distribution.

Finally, let $A(t)$ and $D_i(t)$ denote the number of arrivals and station $i$ departures, respectively, on $[0,t]$. If an arrival occurs at time $t$, we let $U(t)$ be the interarrival time of the subsequent customer, and if a station $i$ departure occurs at time $t$, we let $S_i(t)$ be the service time of the next customer to be served. The BAR for $Z$ is proved analogously to Lemma 1.

LEMMA 8. *Initialize $Z(0) \sim Z$. If $\mathbb{E}|f(Z)| < \infty$ and, $Z(0)$-almost surely, $df(Z(t))/dt$ exists for almost all $t$, then, provided that all expectations are well defined,*

$$\begin{aligned}
&-\mathbb{E}(\partial_{r_a} f(Z)) - \sum_{i=1}^{2} \mathbb{E}(1(X_i > 0)\partial_{r_{s,i}} f(Z)) \\
&+ \mathbb{E}\int_0^1 \Delta f(Z(t-))dA(t) + \sum_{i=1}^{2} \mathbb{E}\int_0^1 \Delta f(Z(t-))dD_i(t) = 0. \qquad (25)
\end{aligned}$$

Define the compensated queue-length vector $\widetilde{X}(t) = (\widetilde{X}_1(t), \widetilde{X}_2(t))$ as

$$\widetilde{X}_1(t) = X_1(t) - \delta_1 \lambda R_a(t) + \delta_1 \mu_1 R_{s,1}(t), \quad \widetilde{X}_2(t) = X_2(t) - \delta_2 \mu_1 R_{s,1}(t) + \delta_2 \mu_2 R_{s,2}(t),$$

and let $\widetilde{X}$ have its stationary distribution. The BAR for $\widetilde{X}$ is

$$\begin{aligned}
&\delta_1 \mathbb{E}\big((\lambda - \mu_1 1(Q_1 > 0))\partial_1 f(\widetilde{X})\big) + \delta_2 \mathbb{E}\big((\mu_1 1(Q_1 > 0) - \mu_2 1(Q_2 > 0))\partial_2 f(\widetilde{X})\big) \\
&+ \mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dA(t) + \sum_{i=1}^{2} \mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dD_i(t) = 0,
\end{aligned}$$

where $\partial_i f(x) = df(x)/dx_i$. Omitting the bulky higher-order terms, one can repeat the arguments of Propositions 1 and 2 to show that

$$\delta_1 \mathbb{E}\big((\lambda - \mu_1 1(Q_1 > 0))\partial_1 f(\widetilde{X})\big) + \delta_2 \mathbb{E}\big((\mu_1 1(Q_1 > 0) - \mu_2 1(Q_2 > 0))\partial_2 f(\widetilde{X})\big)$$

$$\approx -\mu_1\delta_1^2\mathbb{E}\partial_1 f(X) + \delta_2(\mu_1\delta_1 - \mu_2\delta_2)\mathbb{E}\partial_2 f(X)$$
$$+ \mu_1\big(\delta_1 1(Q_1=0)\partial_1 f(X) - \delta_2 1(Q_1=0)\partial_2 f(X)\big) + \mu_2\delta_2 1(Q_2=0)\partial_2 f(X),$$

and

$$\begin{aligned}
&\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dA(t) \approx \frac{1}{2}\delta_1^2\lambda c_U^2\mathbb{E}\partial_{11} f(X),\\
&\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dD_1(t) \approx \frac{1}{2}\mu_1 c_{S,1}^2\mathbb{E}\big(\delta_1^2\partial_{11} f(X) - 2\delta_1\delta_2\partial_{12} f(X) + \delta_2^2\partial_{22} f(X)\big),\\
&\mathbb{E}\int_0^1 \Delta f(\widetilde{X}(t-))dD_2(t) \approx \frac{1}{2}\delta_2^2\mu_2 c_{S,2}^2\mathbb{E}\partial_{22} f(X),
\end{aligned}$$

suggesting a diffusion generator of the form

$$\begin{aligned}
G_Y f(x) = &-\mu_1\delta_1^2\partial_1 f(x) + \delta_2(\mu_1\delta_1 - \mu_2\delta_2)\partial_2 f(x) + \frac{1}{2}\delta_1^2\big(\lambda c_U^2 + \mu_1 c_{S,1}^2\big)\partial_{11} f(x)\\
&- \delta_1\delta_2\mu_1 c_{S,1}^2\partial_{12} f(x) + \frac{1}{2}\delta_2^2\big(\mu_1 c_{S,1}^2 + \mu_2 c_{S,2}^2\big)\partial_{22} f(x),\\
&+ \mu_1(1(x_1=0)(\delta_1\partial_1 f(x) - \delta_2\partial_2 f(x)) + \mu_2\delta_2 1(x_2=0)\partial_2 f(x), \quad x\in\mathbb{R}_+^2.
\end{aligned}$$

This generator corresponds to a two-dimensional RBM $\{Y(t)\in\mathbb{R}_+^2 : t\geq 0\}$ on the nonnegative orthant defined as follows. Set

$$\mu = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \quad \Sigma = \begin{pmatrix}\lambda c_U^2 + \mu_1 c_{S,1}^2 & -\mu_1 c_{S,1}^2\\ -\mu_1 c_{S,1}^2 & \mu_1 c_{S,1}^2 + \mu_2 c_{S,2}^2\end{pmatrix}, \quad R = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix},$$

let $\{\xi(t) : t\geq 0\}$ be a two-dimensional Brownian motion with drift $b=-R\mu$ and covariance matrix $\Sigma$, and let

$$Y(t) = \begin{pmatrix}\delta_1 & 0\\ 0 & \delta_2\end{pmatrix}\widetilde{Y}(t),$$

where

$$\widetilde{Y}(t) = \xi(t) + RI(t), \quad t \geq 0,$$

and $I : R \to \mathbb{R}^2$ is the unique nondecreasing process with $I(0) = 0$ and

$$\int_0^\infty \widetilde{Y}_i(t) dI_i(t) = 0, \quad i = 1, 2.$$

For the existence and uniqueness of $\{\widetilde{Y}(t) : t \geq 0\}$ see Harrison and Reiman (1981).

### 4.1. The Stein method step

Proceeding analogously to Section 2.1, fix $h : \mathbb{R}^2 \to \mathbb{R}$ with $\mathbb{E}|h(Y)| < \infty$, where $Y$ has the stationary distribution of $\{Y(t) : t \geq 0\}$, and consider the Poisson equation for this two-dimensional RBM. Namely,

$$G_Y f_h(x) = \mathbb{E}h(Y) - h(x), \qquad x \in \mathbb{R}^2_+,$$

$$\delta_1 \partial_1 f(x) - \delta_2 \partial_2 f(x) = 0, \qquad x = (0, x_2) \in \mathbb{R}^2_+, \tag{26}$$

$$\partial_2 f(x) = 0, \qquad x = (x_1, 0) \in \mathbb{R}^2_+. \tag{27}$$

This is known as an oblique derivative problem (Lieberman (2013)) with (26) and (27) arising from the reflection structure of the RBM, which is driven by the matrix $R$. To bound the diffusion approximation error for the tandem queue, we require bounds on the partial derivatives, up to the third order, of $f_h(x)$. These Stein factor bounds have not been established and remain an open problem.

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