Local hedging approximately solves Pandora's box problems with nonobligatory inspection

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Abstract

We consider search problems with nonobligatory inspection and single-item or combinatorial selection. A decision maker is presented with a number of items, each of which contains an unknown price, and can pay an inspection cost to observe the item's price before selecting it. Under single-item selection, the decision maker must select one item; under combinatorial selection, the decision maker must select a set of items that satisfies certain constraints. In our nonobligatory inspection setting, the decision maker can select items without first inspecting them. It is well-known that search with nonobligatory inspection is harder than the well-studied obligatory inspection case, for which the optimal policy for single-item selection (Weitzman, 1979) and approximation algorithms for combinatorial selection (Singla, 2018) are known.

We introduce a technique, *local hedging*, for constructing policies with good approximation ratios in the nonobligatory inspection setting. Local hedging transforms policies for the obligatory inspection setting into policies for the nonobligatory inspection setting, at the cost of an extra factor in the approximation ratio. The factor is instance-dependent but is at most 4/3. We thus obtain the first approximation algorithms for a variety of combinatorial selection problems, including matroid basis, matching, and facility location.

Contents

1 Introduction

In the now classic *Pandora's box* problem of Weitzman (1979), a decision maker (henceforth, DM) possesses N items, each of which contains an unknown price. Items can be inspected sequentially, at a cost, in any order, and after search, the DM selects one item (or, in generalizations to be discussed shortly, a set of items). The original Pandora's box model has *obligatory inspection*: when search stops, the DM must select a previously inspected item. In contrast, we consider the problem with *nonobligatory inspection* (Guha et al., 2008; Doval, 2018; Beyhaghi and Kleinberg, 2019; Beyhaghi and Cai, 2023; Fu et al., 2023): when search stops, the DM may select any item, regardless of whether it has been inspected.

The uncertainty in Pandora's box problems is stochastic. Specifically, each item n has a known inspection cost c_n, which is deterministic; and a hidden quantity V_n, which is stochastic, drawn from a distribution known to the DM but with unknown realization. Paying the item's inspection cost c_n reveals the realization of V_n. One can consider expected reward maximization or expected cost minimization versions of this problem; we consider the cost-minimization version, and thus call V_n the item's *hidden price*. That is, the DM must pay the hidden price of the item(s) they select.

Remarkably, the original obligatory inspection Pandora's box model admits an elegant optimal solution. Items are assigned *reservation prices* (see (RP) in Section 3); the DM inspects items in increasing order of their reservation prices; search stops when the minimum inspected price is lower than the minimum reservation price among uninspected items. Each item's reservation price can be computed "locally", with knowledge of that item's hidden price distribution and inspection costs alone. Weitzman's policy thus breaks the curse of dimensionality, as the computation required scales only linearly with the number of boxes.

The simplicity of Weitzman's policy has another benefit: *compositionality*. Specifically, Singla (2018) shows that one can compose Weitzman's policy with (roughly) any greedy algorithm to solve *combinatorial* Pandora's box problems. These are combinatorial optimization problems where deterministic weights are replaced by inspectable items. For instance, the Pandora's box version of minimum spanning tree would have each item sit on an edge of a graph, each containing a hidden price, and the DM's goal is to select a set of items that forms a spanning tree while minimizing the expected sum of edge prices and inspection costs. Singla (2018) shows that if the greedy algorithm is a β-approximation for the deterministic version of the problem, then composing it (in an appropriate sense) with Weitzman's reservation price policy yields a β-approximation for the Pandora's box version of the problem with obligatory inspection.

Unfortunately, the nonobligatory inspection variant of the Pandora's box problem does not inherit the simple optimal policy of its obligatory inspection cousin (Doval, 2018). In particular, finding the optimal policy is known to be NP-hard (Fu et al., 2023). While recent work has shown the single-item selection problem admits a PTAS (Fu et al., 2023; Beyhaghi and Cai, 2023), the algorithm lacks the simplicity and compositionality of Weitzman's policy for obligatory inspection. In particular, there are no approximation algorithms for combinatorial Pandora's box problems with nonobligatory inspection.

In this paper, we define a class of policies, called *local hedging*, for Pandora's box problems with nonobligatory inspection. Local hedging is a *randomized* variant of Weitzman's reservation price policy. Crucially, local hedging maintains both the simplicity and compositionality of Weitzman's policy. In the traditional single-item selection problem, local hedging yields a $\frac{4}{3}$-approximation. In the combinatorial setting, we show that for any greedy algorithm for which Singla's result

(Singla, 2018) yields a β-approximation, composing local hedging with that greedy algorithm yields a $\frac{4}{3}\beta$-approximation. Local hedging thus gives the *first approximation algorithms* for combinatorial Pandora's box problems with nonobligatory inspection.

1.1 Key ideas

At a high level, the complexity of the nonobligatory inspection Pandora's box problem comes from the fact that each uninspected item has two available actions: the DM can inspect it, or the DM can select it without inspection. A natural idea to tame this complexity is to adopt what Beyhaghi and Kleinberg (2019) call a *committing policy*. Such a policy labels each item as either obligatory-inspection or non-inspection, then obeys these labels. That is:

- By labeling an item as obligatory-inspection, the DM commits to never selecting it unless they inspect it first.
- By labeling an item as non-inspection, the DM commits to never inspecting it.

The key idea is that any policy obeying these labels is essentially solving a obligatory inspection problem,[1] so results of Weitzman (1979) and Singla (2018) can be applied.

It has been shown that in the reward maximization version of single-item selection under nonobligatory inspection, there is a committing policy that achieves a constant-factor approximation (Beyhaghi and Kleinberg, 2019; Guha et al., 2008). However, this result is nonconstructive. The resulting algorithm simply evaluates the expected value of all committing policies, then chooses the best one. This is feasible for single-item selection: it is clear that one should never label more than one item as non-inspection, so there are only $N + 1$ committing policies to consider. But it is not feasible for combinatorial selection problems. For instance, for k-item selection, there are $O(N^k)$ committing policies to consider.

Local hedging is a *randomized* committing policy with an *explicit construction*. Based on each item's inspection cost, mean hidden price, and reservation price, local hedging determines an item-specific hedging probability p_n. This is "local" in the sense that, like Weitzman's reservation prices, the hedging probabilities does not depend on any other items' parameters. Under local hedging, the DM simply labels item n as obligatory inspection with probability p_n independently across items. The DM then applies the obligatory inspection policy of Weitzman (1979) (for single-item selection) or Singla (2018) (for combinatorial selection). We show in Theorems 4.4 and 5.4 that, roughly speaking, this randomized commitment inflates the approximation ratio by at most $\frac{4}{3}$. The $\frac{4}{3}$ is actually a conservative bound on an instance-dependent factor, explained in more detail below.

There are two main technical challenges to defining and analyzing local hedging:

- Deciding each item's hedging probability.
- Comparing local hedging's expected cost to that of the intractable optimal policy.

We solve both problems by defining a new notion, which we call *local approximation* (Definition 4.2). Roughly speaking, a given hedging probability yields a local α-approximation for a given item if, in "any context", randomly committing using that hedging probability is no worse than inflating the item's inspection cost and hidden price by a factor of α. A priori, the "any context" requirement may seem difficult to satisfy. One of our key insights is that for the purposes of minimizing expected cost,

[1] If item n is labeled non-inspection, it is equivalent to a obligatory inspection item with inspection cost 0 and hidden price that is deterministically $\mathbf{E}[V_n]$, because the objective is *expected* cost minimization.

it suffices to consider only the simplest nontrivial contexts: those in which the DM must choose between selecting the given item and selecting a single outside option of known deterministic cost.

With the notion of local approximation in hand, our single-item selection results amount to three main steps.

- Prove a tractable lower bound on the optimal expected cost (Theorem 3.4).[2]
- Prove that any item, no matter its inspection cost or hidden price distribution, admits a local α-approximation for some $\alpha \leq \frac{4}{3}$.
- Prove that if all items admit a local α-approximation, then local hedging with the associated hedging achieves expected cost at most α times the aforementioned lower bound (Theorem 4.4).

The end result is that local hedging is a $(\max_{n \in [N]} \alpha_n)$-approximation for single-item selection, where α_n is the best local approximation ratio achievable for item n. Our combinatorial results follow the same outline. In fact, only the first and third steps need to change (Theorems 5.2 and 5.4).

1.2 Related Literature

Because of its relevance to fundamental applications ranging from innovation to search in online markets, Weitzman (1979) initiated a vast literature studying the problem of sequential inspection in Economics, Marketing, Computer Science, and Operations Research (see the surveys by Armstrong (2017) and Ursu et al. (2023), and Derakhshan et al. (2022) for a recent application to product rankings). Since its inception, several variations to the model have been considered. For instance, Klabjan et al. (2014) consider the case of multiple attributes; Chawla et al. (2020) and Gergatsouli and Tzamos (2023) consider the case in which the boxes' contents are correlated; Singla (2018), Boodaghians et al. (2020), Gupta et al. (2019), Gergatsouli and Tzamos (2022), and Aminian et al. (2023) study sequential inspection under various constraints; Hoefer et al. (2021) and Bhaskara et al. (2022) consider the case in which each box can be probed multiple times. Of particular relevance to our work is the aforementioned result of Singla (2018), who considers selecting multiple items with combinatorial constraints on admissible selection sets. This result was later generalized by Gupta et al. (2019) to models where item inspection is not an atomic operation, but rather a multi-stage process (see also Kleinberg et al. (2016, Appendix G) and Aouad et al. (2020)).

Out of all these variations, the case of nonobligatory inspection has recently captured the attention of researchers in these areas, starting from the work of Guha et al. (2008) and Doval (2018). Whereas Doval (2018) focuses on properties of optimal policies, Guha et al. (2008) provides a 0.8-approximation algorithm. Beyhaghi and Kleinberg (2019) show that a class of committing policies provides a 0.63-approximation algorithm. Fu et al. (2023) show that computing an optimal policy is NP-Hard. Relying on a new structural property, Fu et al. (2023) and Beyhaghi and Cai (2023) provide polynomial time approximation schemes that for any $\varepsilon > 0$ compute policies with an expected payoff of at least $(1 - \varepsilon)$ of the optimal.

Our contribution to the nonobligatory inspection literature is threefold. First, whereas most of the literature focuses on the single-item-rewards case, our local-hedging policy accommodates both costs and combinatorial selection as in Singla (2018). Second, whereas the aforementioned

[2] As we discuss in Section 1.2, the single-item version of our lower bound is an instance of the "Whittle's integral" bound for bandit superprocesses (Whittle, 1980; Brown and Smith, 2013; Aouad et al., 2020). But our extension to the combinatorial setting (Theorem 5.2) is, to the best of our knowledge, novel.

PTAS results rely on structural properties of the optimal policy for single-item selection under nonobligatory inspection, local hedging only relies on the properties of the much simpler obligatory inspection case. It follows that contrary to the existing results, local hedging relies only on calculating two numbers for each box: a reservation value and a hedging probability. Finally, as we discuss in Section 6, local hedging could potentially extend beyond the Pandora's box model with nonobligatory inspection to other so-called *Markovian bandit superprocess* problems (Gittins et al., 2011; Whittle, 1980; Glazebrook, 1982). Specifically, our notion of local approximation (Definition 4.2), when appropriately generalized, is a relaxation of a condition from the superprocess literature (Glazebrook, 1982), and we expect our main lower and upper bound theorems to similarly generalize.

In terms of technical approach, our work is closest to that of Beyhaghi and Kleinberg (2019). As previously noted, local hedging is a randomization over the class of committing policies considered by Beyhaghi and Kleinberg (2019). Moreover, Beyhaghi and Kleinberg (2019) also prove a lower bound on the expected optimal cost of single-item selection. Our bound (Theorem 3.4) is always less than theirs, but it is more explicit. See Appendix B for a detailed comparison. We emphasize, however, that Beyhaghi and Kleinberg (2019) do not consider the combinatorial case.

Our single-item lower bound is an instance of a "Whittle's integral" bound, which is a method of bounding the optimal performance in bandit superprocess problems (Whittle, 1980; Brown and Smith, 2013; Aouad et al., 2020). The closest to ours is an upper bound in Aouad et al. (2020, Lemma 1), which is an upper bound for a variant of the Pandora's box problem with multiple stages of inspection for each item. Their bound is a translation of a result of Brown and Smith (2013, Proposition 4.2) (who build upon Whittle (1980, Section 5)) from the discounted bandit superprocess setting to an undiscounted Pandora's-box-type setting. Our single-item lower bound is another translation of the same, with some minor differences (e.g. we treat minimization vs. prior work treating maximization). The main novelty of our bound is extending the Whittle's integral method to the combinatorial setting. A secondary novelty is that we give the bound an elegant probabilistic interpretation in the "capped value" style that is a hallmark of the Pandora's box setting (Kleinberg et al., 2016; Beyhaghi and Kleinberg, 2019).

1.3 Organization

The rest of the paper is organized as follows. Section 2 introduces the single-item selection model that we use to introduce our results. Section 3 proves a lower bound on the expected cost of the optimal policy (Theorem 3.4). Section 4 introduces local hedging and gives an upper bound on its expected cost (Theorem 4.4). Section 5 covers the combinatorial case, introducing the combinatorial-inspection model and extending both our lower and upper bounds (Theorems 5.2 and 5.4). Section 6 concludes with a discussion of how local hedging might extend to settings with reward maximization (we consider cost minimization throughout), as well as settings with more general Markovian bandit superprocesses.

2 Model

We introduce the single-item selection model, which we then use to state our main results. Though our results hold for more general combinatorial selection problems, the single-item selection model allows us to present the most streamlined version of our results that best emphasizes the key intuition. See Section 5 for the generalization to combinatorial selection.

Single-item nonobligatory inspection A decision maker (henceforth, DM) possesses N items, indexed by $n \in [N] = \{1, \ldots, N\}$. Each item n contains an unknown price, V_n, distributed according to G_n, with mean value μ_n. The distributions $\{G_n : n \in [N]\}$ are independent. To observe item n's price, the DM must pay an inspection cost $c_n \geq 0$.

We refer to the tuple $\{(G_n, c_n) : n \in [N]\}$ as an *instance*. To clarify, the DM knows the instance, but does not know the realization of each $V_n \sim G_n$ until after inspecting item n.

The DM's goal is to adaptively inspect a set of items and select one item while minimizing the expected total cost, where the total cost C is given by:

$$C = \sum_{n \in [N]} (\mathbb{S}_n V_n + \mathbb{I}_n c_n) \tag{TC}$$

In TC, $\mathbb{S}_n$ and $\mathbb{I}_n$ are the indicators that n is selected and inspected, respectively.[3] Because the DM must select an item, $\sum_{n \in [N]} \mathbb{S}_n = 1$. Note, however, that we do not impose that $\mathbb{S}_n \leq \mathbb{I}_n$ and hence, we allow the DM to select an item without having first inspected its contents. In other words, our model corresponds to Pandora's box with nonobligatory inspection (Guha et al., 2008; Doval, 2018).

Letting Π^{NOI} denote the set of all (adaptive) policies in the nonobligatory inspection problem, the DM must choose a policy $\pi \in \Pi^{\mathsf{NOI}}$ to minimize $\mathbf{E}[C^\pi]$, where we let C^π denote the cost induced by policy π. We additionally denote the optimal expected cost by

$$\bar{C}^{\mathsf{NOI}} = \min_{\pi \in \Pi^{\mathsf{NOI}}} \mathbf{E}[C^\pi]. \tag{OPT}$$

Obligatory inspection If, instead, we assume that the DM can only take an item it has already inspected (that is, $\mathbb{S}_n \leq \mathbb{I}_n$), the above model corresponds to the Pandora's box model in Weitzman (1979). In what follows, we refer to this model as the obligatory inspection model and we let Π^{OI} denote the set of policies available to the DM under obligatory inspection. Mirroring (OPT), we let $\bar{C}^{\mathsf{OI}} = \min_{\pi \in \Pi^{\mathsf{OI}}} \mathbf{E}[C^\pi]$.

Remark 2.1 (Notational conventions). We collect in one place our notational conventions. To indicate that we refer to the nonobligatory or obligatory inspection models, we label variables with NOI or OI. For instance, Π^{NOI} and Π^{OI} denote the admissible policies under the nonobligatory inspection and obligatory inspection models, respectively. While C^π and other notations defined later depend on the instance $\{(G_n, c_n) : n \in [N]\}$, we suppress this dependence from on our notation.

3 A lower bound on the optimal cost

In this section, we state and prove Theorem 3.4, which provides a lower bound on the cost of the optimal policy for the nonobligatory inspection problem. Our lower bound is expressed in terms of each item's *surrogate prices*, defined in Definition 3.2. In order to define surrogate prices, it is helpful to consider a special case of the problem in which the DM is choosing between one item

[3] We follow the notation in Kleinberg et al. (2016) and Beyhaghi and Kleinberg (2019).

and an outside option of known value. We begin with this special case, which we call the *one-item subproblem.*

Throughout, we focus primarily on nonobligatory inspection, but we discuss obligatory inspection when explaining important background, or when it will be important for our later analysis. All of the results for obligatory inspection are standard. One of results for nonobligatory inspection (Lemma 3.1) is due to Doval (2018). The other obligatory inspection results are novel.

3.1 The one-item subproblem

Consider the case in which the DM has a single item and an outside option of known value, $r \in \mathbb{R}$. One can think of the outside option as a second item that has inspection cost 0 and deterministic hidden price r. In what follows, to simplify notation, we omit the index n from our notation: the single item has hidden price $V \sim G$ and inspection cost c.

We denote the expected cost of using a policy π for the one-item subproblem by $C^{\pi}_{\text{item}}(r)$, and let $\bar{C}^{\text{NOI}}_{\text{item}}(r) = \min_{\pi \in \Pi^{\text{NOI}}} \mathbf{E}[C^{\pi}_{\text{item}}(r)]$. With that said, for any given one-item subproblem, there are only three policies that could possibly be optimal:

(a) Select the outside option. This costs r.
(b) Inspect the item, then select the better between the item's hidden price V and the outside option r. This costs $c + \mathbf{E}[\min\{V, r\}]$ in expectation.
(c) Select the item without inspection. This costs μ in expectation.

The optimal cost for the one-item subproblem is achieved by picking the best among these three, so

$$\bar{C}^{\text{NOI}}_{\text{item}}(r) = \min_{\pi \in \Pi^{\text{NOI}}} \mathbf{E}[C^{\pi}_{\text{item}}(r)] = \min\big\{c + \mathbf{E}[\min\{V, r\}], r, \mu\big\}. \tag{3.1}$$

It is intuitive that (a) is best for small r and that (c) is best for large r, with (b) best at intermediate values. Doval (2018, Proposition 0) formalizes this intuition, characterizing the values of r for which each is optimal (see also Guha et al., 2008). We restate the result below, adapting it from rewards to costs.

Lemma 3.1 (Proposition 0 in Doval, 2018)**.** *Define an item's* reservation price u^{rsv} *and* backup price u^{bkp} *implicitly as follows:*

$$\mathbf{E}_G[(u^{\text{rsv}} - V)^+] = c, \tag{RP}$$

$$\mathbf{E}_G[(V - u^{\text{bkp}})^+] = c. \tag{BP}$$

For all $r \in \mathbb{R}$, the optimal policy in the one-item subproblem is as follows. If $u^{\text{rsv}} \geq u^{\text{bkp}}$, then the DM selects the item without inspection. Instead, if $u^{\text{rsv}} < u^{\text{bkp}}$, the DM

- *takes the outside option r if $r \leq u^{\text{rsv}}$,*
- *selects the item without inspection if $r \geq u^{\text{bkp}}$, and*
- *otherwise inspects the item and selects whatever is best between the item's price V and the outside option r.*

The item's reservation price is the value of the outside option that makes the DM indifferent between taking the outside option and inspecting the item (cf. Weitzman, 1979). Similarly, the item's backup price is the value of the outside option that makes the DM indifferent between inspecting

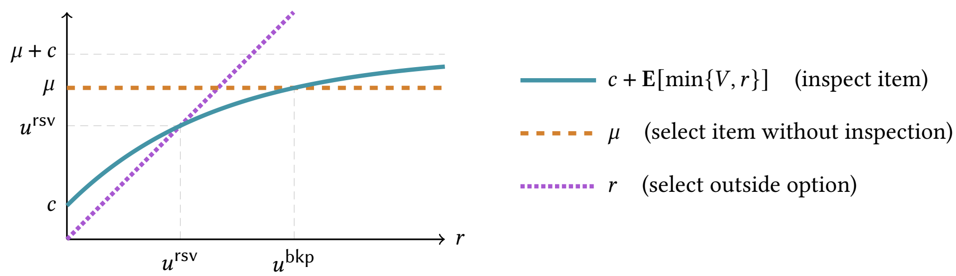


Figure 3.1: Illustration of the expected cost in the one-item subproblem given each of the three possible first actions. The reservation and backup prices are the values of r that cause indifference between inspecting and the other two actions.

the item and selecting it without inspection (cf. Doval, 2018). See Figure 3.1 for an illustration. The optimal policy for the one-item subproblem follows from these indifference properties.

One can also study the one-item subproblem with obligatory inspection. We write $\bar{C}^{\text{OI}}_{\text{item}}(r) = \min_{\pi \in \Pi^{\text{OI}}} \mathbf{E}[C^{\pi}_{\text{item}}(r)]$ for the optimal expected cost in this setting. It is a standard result that $\bar{C}^{\text{OI}}_{\text{item}}(r) = \min\{c + \mathbf{E}[\min\{V, r], r\}$, analogous to (3.1).

3.2 Surrogate prices

Having described the one-item subproblem, it remains to relate its properties back to the full single-item selection problem with multiple items. We do so by way of *surrogate prices*, defined below. Surrogate prices give a convenient way of characterizing the optimal cost not just in the one-item subproblem (see Lemma 3.3), but also the full single-item selection problem (see Theorem 3.4).

Definition 3.2 (Surrogate prices)**.** An item's *obligatory inspection surrogate price* (henceforth, OI-surrogate price), denoted by W^{OI}, is the random variable

$$W^{\text{OI}} = \max\{V, u^{\text{rsv}}\}. \tag{OI-S}$$

An item's *nonobligatory inspection surrogate price* (henceforth, NOI-surrogate price), denoted by W^{NOI}, is the random variable

$$W^{\text{NOI}} = \begin{cases} \min\{W^{\text{OI}}, u^{\text{bkp}}\} & \text{if } u^{\text{rsv}} < u^{\text{bkp}} \\ \mu & \text{if } u^{\text{rsv}} \geq u^{\text{bkp}}. \end{cases} \tag{NOI-S}$$

The item's OI-surrogate price captures that in order to obtain the price V, the DM must first inspect the item. By inflating the item's cost up to its reservation price, the OI-surrogate price W^{OI}_n internalizes the item's inspection cost. The NOI-surrogate price additionally deflates the item's price down to its backup price u^{bkp}. This internalizes that under nonobligatory inspection, the inspection cost may not be paid after all.

Because an item's surrogate price internalizes the inspection cost, we may express the expected optimal cost of the one-item subproblem using the expectation of a *one-shot* choice between the surrogate price and the outside option. This is a standard result in the obligatory inspection setting. We extend it to the nonobligatory inspection setting in Lemma 3.3 below.

Lemma 3.3 (Surrogate prices solve one-item nonobligatory inspection)**.** *For all $r \in \mathbb{R}$,*

$$\bar{C}^{\mathsf{OI}}_{\mathsf{item}}(r) = \min\{c + \mathbf{E}[\min\{V, r\}, r]\} = \mathbf{E}[\min\{W^{\mathsf{OI}}, r\}],$$
$$\bar{C}^{\mathsf{NOI}}_{\mathsf{item}}(r) = \min\{c + \mathbf{E}[\min\{V, r\}, r, \mu]\} = \mathbf{E}[\min\{W^{\mathsf{NOI}}, r\}].$$

Proof sketch. The obligatory inspection result is standard, so we focus on nonobligatory inspection. One way to see the result is to observe that W^{NOI} is defined in (NOI-S) such that

$$\mathbf{P}[W^{\mathsf{NOI}} > r] = \frac{\mathrm{d}}{\mathrm{d}r}\bar{C}^{\mathsf{NOI}}_{\mathsf{item}}(r),$$

from which the result follows by integration. See Appendix A for the complete proof. □

3.3 Optimal cost lower bound

Having used the one-item subproblem to introduce NOI-surrogate prices, we are now ready to state our first main result for the full single-item selection problem.

Theorem 3.4 (Single-item selection lower bound)**.** *In nonobligatory inspection single-item selection, the optimal policy's expected total cost satisfies*

$$\bar{C}^{\mathsf{NOI}} = \min_{\pi \in \Pi^{\mathsf{NOI}}} \mathbf{E}[C^{\pi}] \geq \mathbf{E}\left[\min_{n \in [N]} W^{\mathsf{NOI}}_n\right]. \tag{LB-OPT}$$

Proof sketch. Consider an arbitrary policy π for the DM. It suffices to define a submartingale that is $\mathbf{E}\big[\min_{n \in [N]} W^{\mathsf{NOI}}_n\big]$ at time 0 and is C^{π} when the policy selects an item. The submartingale is

$$K(t) = C(t) + \mathbf{E}\left[\min_{n \in [N]} W_n(t) \,\middle|\, \mathcal{I}(t)\right],$$

where

$$C(t) = \text{total inspection and selection cost paid by } \pi \text{ during } \{0, \dots, t-1\},$$
$$W_n(t) = \begin{cases} 0 & \text{if any item is selected by } \pi \text{ during } \{0, \dots, t-1\} \\ V_n & \text{if } n \text{ is inspected, but not selected, by } \pi \text{ during } \{0, \dots, t-1\} \\ W^{\mathsf{NOI}}_n & \text{otherwise,} \end{cases}$$
$$\mathcal{I}(t) = \text{information } \pi \text{ gains from inspections during } \{0, \dots, t-1\}.$$

That is, the key idea is to define a time-dependent surrogate price, which is the NOI-surrogate price prior to inspection and the hidden price thereafter. Roughly speaking, $K(t)$ is a submartingale because NOI-surrogate-prices internalize inspection costs, but they do so "optimistically", in some sense assuming that an inspected item can later be selected without inspection. We formally verify $K(t)$ is a submartingale using Lemma 3.1 and some straightforward computation. See Appendix A for the complete proof. □

Theorem 3.4 states that the expected value of the one-shot problem in which the DM chooses the best NOI-surrogate price is a lower bound for the optimal cost under nonobligatory inspection. In contrast to Lemma 3.3's equality, Theorem 3.4 is an inequality: the value of the one-shot problem may not be feasible in the nonobligatory inspection problem. This is because, roughly speaking, attempting to achieve the value of the one-shot problem may effectively ask the DM to first inspect a given item, but then later select it without inspection. The value of Theorem 3.4 is that the right hand side of LB-OPT can be computed directly from one-item subproblems, whereas it is well understood that computing the optimal cost in the nonobligatory inspection problem is intractable.

Anticipating the analysis of the combinatorial case in Section 5, we note that Theorem 3.4 extends to the combinatorial case (see Theorem 5.2).

Remark 3.5. Beyhaghi and Kleinberg (2019, Lemma 16) prove a result that is similar to Theorem 3.4, giving a bound on $\mathbf{E}[C^\pi]$ for any policy $\pi \in \Pi^{\mathsf{NOI}}$. Their bound is actually tighter than ours, but it is less explicit, because their bound expression also depends on π. We state their bound and discuss it in more detail in Appendix B, including an alternate proof of Theorem 3.4 by way of their bound.

Obligatory inspection It is also useful to contrast Theorem 3.4 with the analogous result for obligatory inspection, which involves the items' OI-surrogate prices. Lemma 3.3 already suggests that in the one-item subproblem with obligatory inspection, the OI-surrogate price summarizes the value of the optimal policy. In contrast to the case of nonobligatory inspection, this result extends to any number of items, as stated in Proposition 3.6 below.

Proposition 3.6 (Corollary 3 in (Beyhaghi and Kleinberg, 2019))**.** *In obligatory inspection single-item selection, the optimal policy's expected total cost is*

$$\bar{C}^{\mathsf{OI}} = \min_{\pi \in \Pi^{\mathsf{OI}}} \mathbf{E}[C^\pi] = \mathbf{E}\left[\min_{n \in [N]} W_n^{\mathsf{OI}}\right].$$

In other words, despite the adaptive nature of the problem in Weitzman (1979), its value *can* be obtained in a one shot problem, in which the DM picks the item with the lowest OI-surrogate price. In fact, not only the values of the two problems coincide, but also the DM selects the same item under both policies (Kleinberg et al., 2016; Beyhaghi and Kleinberg, 2019). In other words, the one-shot problem also describes the item that is *eventually* selected after adaptive inspection in Pandora's box problem.

The contrast between Theorem 3.4 and Proposition 3.6 leaves open the question of whether a *feasible* policy for the nonobligatory inspection case exists that provides a reasonable upper bound for the optimal cost, while at the same time inheriting the simplicity of the policy that delivers the optimal cost of Weitzman (1979). As we explain next, our local hedging policy does precisely this.

4 Local hedging

The local hedging policy that we introduce in this section allows us to marry the results in Theorem 3.4 and Proposition 3.6 to provide a lower bound on the optimal cost for nonobligatory inspection. Local hedging inherits the property of obligatory inspection that its value can be calculated from the corresponding surrogate prices. In contrast to the one-shot problem that is defined by the NOI-surrogate prices, the local hedging policy always induces a feasible policy for nonobligatory inspection.

4.1 Local hedging for the one-item subproblem

To define the local hedging policy, we consider first the one-item subproblem (see Section 3.1). In this setting, the local hedging policy is parameterized by a single parameter $p \in [0, 1]$, which we call the item's *hedging probability*. At the beginning, the DM flips a p-weighted coin.

- With probability p, the DM labels the item as *obligatory-inspection*.
- With probability $1 - p$, the DM labels the item as *non-inspection*.

These labels constitute commitments from the DM: a non-inspection item will never be inspected, and a obligatory-inspection item will never be selected without inspection. After making this commitment, the DM takes the optimal action that respects this commitment. That is:

- With probability p, the DM treats the one-item subproblem as having obligatory inspection, choosing between the item and the outside option as in Weitzman (1979). Specifically:
  - If $r \leq u^{\mathrm{rsv}}$, the DM selects the outside option r.
  - If $r > u^{\mathrm{rsv}}$, the DM inspects the item, then selects whatever is best between its price V and the outside option r.
- With probability $1 - p$, the DM selects whatever is best between the outside option r and the item's expected value $\mu = \mathbf{E}[V]$.

Each hedging probability p defines a local hedging policy, denoted $\mathsf{LH}(p)$. Below, we sometimes refer to the *local p-hedging policy* when we want to emphasize the specific hedging probability p.

Like we did for obligatory and nonobligatory inspection, we can define an item's local p-hedging surrogate price.

Definition 4.1 (Local hedging surrogate prices)**.** An item's *local p-hedging surrogate price* (henceforth, LH-surrogate price), denoted by $W^{\mathsf{LH}(p)}$, is the random variable

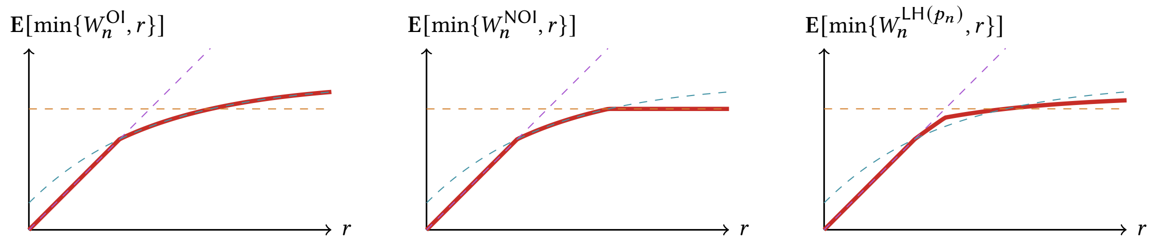
$$W^{\mathsf{LH}(p)} = \begin{cases} W^{\mathsf{OI}} & \text{with probability } p \\ \mu & \text{with probability } 1 - p. \end{cases} \tag{LH-S}$$

See Figure 4.1 for a comparison between the different types of surrogate prices. The definition of the surrogate price makes evident that under local p-hedging, the DM faces the same problem as in Weitzman (1979) with probability p, and with the remaining probability the DM faces an item of known value, μ. Importantly in what follows, the local hedging policy is a policy for the nonobligatory inspection problem that runs by randomizing over policies for the *induced* obligatory inspection problem. In particular, Lemma 3.3 implies

$$\mathbf{E}[C_{\mathsf{item}}^{\mathsf{LH}(p)}(r)] = \mathbf{E}[\min\{W^{\mathsf{LH}(p)}, r\}].$$

4.2 Local approximation

A natural question is whether a hedging probability p exists such that local hedging provides a good approximation of the optimal policy under nonobligatory inspection. We start by defining our notion of approximation in the single-item case.



(a) OI-surrogate price (b) NOI-surrogate price (c) LH-surrogate price($p = \frac{1}{2}$)

Figure 4.1: Illustrations of the different types of surrogate prices.

Definition 4.2 (Local α-approximation)**.** Consider an item with inspection cost c and hidden price $V \sim G$. We say that local p-hedging is a *local α-approximation* for the item if for all $r \in \mathbb{R}$,

$$\mathbf{E}[\min\{W^{\mathsf{LH}(p)}, r\}] \leq \mathbf{E}[\min\{\alpha W^{\mathsf{NOI}}, r\}]. \tag{4.1}$$

We say the item *admits* a local α-approximation if there exists $p \in [0, 1]$ such that local p-hedging is a local α-approximation.

The intuition behind local α-approximation is as follows. Recall from Lemma 3.3 that $\mathbf{E}[\min\{W^{\mathsf{NOI}}, r\}]$ is the optimal cost in the one-item subproblem under nonobligatory inspection. Thus, (4.1) states that in the one-item subproblem, given the choice between

- using local hedging to randomly commit the item to be obligatory-inspection or non-inspection; or
- inflating the item's costs, namely inspection cost and hidden price, by a factor of α;

local hedging is preferable *for all* outside option values r. It is thus conceivable that in the full single-item selection problem, using local hedging for all items is preferable to inflating costs of all items. Indeed, this is the core idea behind our multi-item result (Theorem 4.4).

In the multi-item setting the outside option r is a stand-in for the continuation value after inspecting the current item. The fact that the value of the outside option is not scaled up by α in (4.1) is therefore *crucial*. If we replaced the right-hand side with $\alpha\mathbf{E}[\min\{W^{\mathsf{NOI}}, r\}]$, we would effectively be asking that using local hedging for a single item be preferable to inflating not just that item's costs, but all other items' costs, too.

Of course, defining local approximation is only useful if it is actually achievable. The main result of this section, Theorem 4.3 below, is that all items admit a local $\frac{4}{3}$-approximation or better. We prove Theorem 4.3 in Section 4.4.

Theorem 4.3 (All items admit local approximation)**.** *Consider an item with inspection cost c and hidden price $V \sim G$. Then $\mathsf{LH}(p)$ is a local α-approximation for the item, where*

$$p = \max\left\{\frac{\mu - u^{\mathsf{rsv}}}{\mu - u^{\mathsf{rsv}} + cu^{\mathsf{rsv}}/\mu}, 0\right\}, \qquad \alpha = \max\left\{\frac{\mu - u^{\mathsf{rsv}} + c}{\mu - u^{\mathsf{rsv}} + cu^{\mathsf{rsv}}/\mu}, 1\right\} \leq \frac{4}{3}. \tag{4.2}$$

In particular, all items admit a local $\frac{4}{3}$-approximation.

4.3 Local hedging for single-item selection

So far, we have shown that local hedging delivers on both our desiderata for the case in which the DM has one item and an outside option with known value. We now show that local hedging's ability to deliver a local α-approximation for the single item instances is key for it to deliver a α-approximation to the optimal cost under nonobligatory inspection.

Note first that the local hedging policy can easily be extended to the case in which the DM has N items. In this case, the policy is parameterized by a vector of hedging probabilities $\mathbf{p} = (p_1, \dots, p_N)$, though we leave this vector implicit in our notation, denoting the policy as simply LH. At the beginning, the DM independently flips N coins, each with its own bias p_n. The DM is then faced with an instance of Pandora's box problem, in which the DM can inspect those items that are labeled obligatory-inspection, and conditional on stopping the DM can obtain whatever is best between the already inspected prices and the minimum expected value amongst those items labeled non-inspection.

Theorem 4.4 (Local hedging solves nonobligatory inspection)**.** *Consider a single-item selection problem with nonobligatory inspection. If every item admits a local α-approximation, then by using the corresponding hedging probabilities, local hedging is an α-approximation for single-item selection:*

$$\mathbf{E}[C^{\mathsf{LH}}] = \mathbf{E}\left[\min_{n \in [N]} W_n^{\mathsf{LH}(p_n)}\right] \leq \alpha \bar{C}^{\mathsf{NOI}}. \tag{4.3}$$

In particular, there exist hedging probabilities such that local hedging is a $\frac{4}{3}$-approximation.

Theorem 4.4 states that local hedging satisfies our two desiderata in any instance of Pandora's box with nonobligatory inspection. Indeed, the equality in (4.3) states that the cost of the local hedging policy can be obtained from the LH-surrogate prices. Moreover, the inequality in (4.3) states that local hedging provides a $\frac{4}{3}$-approximation (or better) to the optimal cost under nonobligatory inspection.

Proof of Theorem 4.4. Fix α, and let p_n be the hedging probability that achieves a local α-approximation for item n. The equality $\mathbf{E}[C^{\mathsf{LH}}] = \mathbf{E}[\min_{n \in [N]} W_n^{\mathsf{LH}(p_n)}]$ follows from Proposition 3.6 and the fact that local hedging treats the problem like a obligatory-inspection problem after making its randomized commitments. Then, using (4.1), we compute

$$\begin{aligned}
\mathbf{E}\big[C^{\mathsf{LH}}\big] &= \mathbf{E}\Big[\min\{W_1^{\mathsf{LH}(p_1)}, W_2^{\mathsf{LH}(p_2)}, \dots, W_N^{\mathsf{LH}(p_N)}\}\Big] \\
&\leq \mathbf{E}\Big[\min\{\alpha W_1^{\mathsf{NOI}}, W_2^{\mathsf{LH}(p_2)}, \dots, W_N^{\mathsf{LH}(p_N)}\}\Big] \\
&\ \vdots \\
&\leq \mathbf{E}\big[\min\{\alpha W_1^{\mathsf{NOI}}, \alpha W_2^{\mathsf{NOI}}, \dots, \alpha W_N^{\mathsf{NOI}}\}\big] = \alpha \mathbf{E}\left[\min_{n \in [N]} W_n^{\mathsf{NOI}}\right] \leq \alpha \bar{C}^{\mathsf{NOI}},
\end{aligned}$$

where the last inequality follows from Theorem 3.4. □

Organization of the rest of this section Section 4.4 proves Theorem 4.3, the local approximation step. Section 4.5 discusses different ways in which our results on local approximation and local hedging are tight. A reader interested in the application of local hedging to combinatorial optimization can skip straight to Section 5 with little loss of continuity.

4.4 Proof of Theorem 4.3

Before proving Theorem 4.3, we prove a lemma that characterizes the local approximation ratio of local p-hedging for arbitrary p. Theorem 4.3 then follows by optimizing p.

Lemma 4.5. *Suppose $u^{\mathrm{rsv}} < u^{\mathrm{bkp}}$. For all $r \in \mathbb{R}$, the local p-hedging policy is a local $\alpha(p)$-approximation, where*

$$\alpha(p) = 1 + \max\left\{\frac{(1-p)(\mu - u^{\mathrm{rsv}})}{u^{\mathrm{rsv}}}, \frac{pc}{\mu}\right\}. \tag{4.4}$$

Before proving Lemma 4.5, let us dissect its statement. The approximation parameter $\alpha(p)$ in (4.4) admits a natural interpretation. The first term in the maximum describes the DM's loss in the event the local p-hedging policy sets the item to be a non-inspection item: The DM loses the option to inspect the item and hence, the item's reservation price.[4] The second term in the maximum describes the DM's loss in the event the local p-hedging policy sets the item to be a obligatory-inspection item: The DM loses the option to take the item without inspection, and this loss is larger the larger the item's inspection cost and/or the smaller the item's expected price are. Thus, (4.4) states that the local p-hedging policy's performance is better the smaller these losses are.

Proof of Lemma 4.5. Throughout the proof, to simplify notation we omit the dependence of $\alpha(p)$ on p and simply denote it by α.

By Definition 4.1 and Lemma 3.3, showing (4.1) amounts to showing the following hold

$$p\,\mathbf{E}[\min\{W^{\mathrm{OI}}, r\}] + (1-p)\min\{\mu, r\} \le \mathbf{E}[\min\{\alpha W^{\mathrm{OI}}, r\}], \tag{4.5}$$

$$p\,\mathbf{E}[\min\{W^{\mathrm{OI}}, r\}] + (1-p)\min\{\mu, r\} \le \alpha\mu. \tag{4.6}$$

We begin by showing (4.6). The left-hand side is increasing[5] in r , but the right-hand side does not depend on r . This means it suffices to show (4.6) in the limit when $r \to \infty$. Using that $\mathbf{E}[W^{\mathrm{OI}}] = c + \mu$ (see Definition 3.2), this reduces to showing

$$pc \le (\alpha - 1)\mu, \tag{4.7}$$

which holds for the value α in (4.4).

We now show (4.5). The main obstacle is giving a tight enough bound on $\mathbf{E}[\min\{\alpha W^{\mathrm{OI}}, r\}]$ in terms of $\mathbf{E}[\min\{W^{\mathrm{OI}}, r\}]$. The key observation is that $\mathbf{E}[\min\{W^{\mathrm{OI}}, r\}]$ is concave and increasing as a function of r. We also know the graph of the function goes through $(u^{\mathrm{rsv}}, u^{\mathrm{rsv}})$ (see Definition 3.2). This means that

$$m(r) = \frac{\mathbf{E}[\min\{W^{\mathrm{OI}}, r\}] - u^{\mathrm{rsv}}}{r - u^{\mathrm{rsv}}}, \tag{4.8}$$

namely the slope of the line passing through $(u^{\mathrm{rsv}}, u^{\mathrm{rsv}})$ and $(r, \mathbf{E}[\min\{W^{\mathrm{OI}}, r\}])$, is decreasing as a function of r . This implies the lower bound

$$\begin{aligned}\mathbf{E}[\min\{\alpha W^{\mathrm{OI}}, r\}] &= \alpha\big(u^{\mathrm{rsv}} + m\big(\tfrac{r}{\alpha}\big)\left(\tfrac{r}{\alpha} - u^{\mathrm{rsv}}\right)\big) \\ &\ge \alpha\big(u^{\mathrm{rsv}} + m(r)\left(\tfrac{r}{\alpha} - u^{\mathrm{rsv}}\right)\big). \end{aligned} \tag{4.9}$$

[4] Recall from the definition of the item's OI-surrogate price that the item's reservation price is the lowest price the DM can hope to obtain taking into account the item's inspection cost.

[5] Here and throughout our proofs, we use increasing and decreasing in their weak senses, i.e., to mean "nondecreasing" and "nonincreasing", respectively.

Applying (4.8) and (4.9) to (4.5), we find (4.5) holds if

$$m(r)\left((\alpha - p)u^{\mathrm{rsv}} - (1-p)r\right) \le (\alpha - p)u^{\mathrm{rsv}} - (1-p)\min\{\mu, r\}.$$

Because $m(r) \le 1$ and $-r \le -\min\{\mu, r\}$, it suffices for the right-hand side to be positive, as then dividing both sides by the right-hand side yields at most 1 on the left-hand side. A sufficient condition for the right-hand side to be positive is

$$(1-p)\mu \le (\alpha - p)u^{\mathrm{rsv}}, \tag{4.10}$$

which holds for the value α in (4.4). □

Proof of Theorem 4.3. If $u^{\mathrm{rsv}} \ge \mu$, then picking $p = 0$ yields $\alpha = 1$, so the interesting case is when $u^{\mathrm{rsv}} < \mu$, or equivalently $u^{\mathrm{rsv}} < u^{\mathrm{bkp}}$. In this case, the value of p that minimizes $\alpha(p)$ from Lemma 4.5 is the value that equalizes the branches of the maximum in (4.4), which is p from (4.2). Computing $\alpha(p)$ then yields the value of α from (4.2).

It remains only to show that the resulting value of α from (4.2) is at most $\frac{4}{3}$. Let $x = u^{\mathrm{rsv}}/\mu$. Because $u^{\mathrm{rsv}} < \mu$, we have $x \in [0, 1)$. This means α is an increasing function of c. From this and the fact that (RP) implies $c < u^{\mathrm{rsv}}$, we compute

$$\alpha = \frac{\mu - u^{\mathrm{rsv}} + c}{\mu - u^{\mathrm{rsv}} + cx} < \frac{\mu - u^{\mathrm{rsv}} + u^{\mathrm{rsv}}}{\mu - u^{\mathrm{rsv}} + u^{\mathrm{rsv}}x} = \frac{1}{1 - x + x^2} \le \frac{4}{3}.$$ □

4.5 Tightness of local hedging's approximation ratio

In this section, we address the question of how tight our bounds on local hedging's approximation ratio are. To frame the discussion more precisely, consider the one-item subproblem with a given item, and define α^* to be the minimum value of α such the item admits a local α-approximation. We answer the following questions:

(Q1) For all distributions G and c, do we have $\min_{p\in[0,1]} \alpha(p) = \alpha^*$? (Answer: no.)
(Q2) For all μ, c, and u^{rsv}, does there exist a distribution G resulting in the given mean and reservation price such that $\min_{p\in[0,1]} \alpha(p) = \alpha^*$? (Answer: yes.)
(Q3) Do there exist c and a distribution G and such that $\alpha^* \approx \frac{4}{3}$? (Answer: yes.)

We expand upon all three answers below. We restrict our attention to the $u^{\mathrm{rsv}} < u^{\mathrm{bkp}}$ case, because when $u^{\mathrm{rsv}} \ge u^{\mathrm{bkp}}$, we simply have $\alpha(0) = \alpha^* = 1$, meaning inspection is never worthwhile. Afterwards, we discuss implications for single-item selection beyond the one-item subproblem.

The answer to (Q1) is no, but only because of one step in the proof. The value of $\alpha(p)$ in (4.4) is the minimum value that satisfies constraints (4.7) and (4.10). So (Q1) reduces to: must α^* satisfy (4.7) and (4.10)? One can show that (4.7) is necessary by looking at the $r \to \infty$ limit, but (4.10) is not. This is because the computation that leads to (4.10) involves (4.9), an inequality which need not be tight.

But the above discussion suggests an answer to (Q2): we have $\alpha^* = \min_{p\in[0,1]} \alpha(p)$ if (4.9) holds with equality. In fact, a closer inspection of the proof reveals that we only need (4.9) to be tight when $r = \mu$. A straightforward computation shows that this holds when

$$V = \begin{cases} 0 & \text{with probability } \frac{c}{u^{\mathrm{rsv}}} \\ \frac{u^{\mathrm{rsv}}\mu}{u^{\mathrm{rsv}} - c} & \text{with probability } \frac{u^{\mathrm{rsv}} - c}{u^{\mathrm{rsv}}}. \end{cases} \tag{4.11}$$

The example in (4.11) also addresses (Q3). If we choose $\mu = 2u^{\mathrm{rsv}} = (2 - \varepsilon)c$ in (4.11), then by the above discussion and (4.2), we have $\alpha^* = \max_{p\in[0,1]} \alpha(p) = \frac{4+\varepsilon}{3+\varepsilon}$, where ε may be arbitrarily small. The fact that the worst-case scenario is when V is a high-variance two-point distribution is unsurprising in light of the results of Beyhaghi and Kleinberg (2019), who use a similar construction to bound the approximation ratio of committing policies in the reward maximization setting.

Finally, let us zoom out from the one-item subproblem to full single-item selection. In some sense, our answer to (Q3) yields single-item selection problems for which local hedging is arbitrarily close to a $\frac{4}{3}$-approximation, because the one-item subproblem is a special case of single-item selection. But this is somewhat unsatisfying given the tractability of the one-item subproblem. We can obtain a more satisfying example by using two items. Item 1 is as in (4.11) and the above answer to (Q3), and item 2 has inspection cost $c_2 = \varepsilon$ and has hidden price equally likely to be $V_2 = \mu_1$ or $V_2 = \mu_1/\varepsilon$. For small enough ε, it is clearly optimal to first inspect item 2, after which the problem becomes a one-item subproblem with item 1 and outside option V_2. Following Lemma 3.1, the optimal policy then either inspects the first item (if $V_2 = \mu_1$) or selects the first item without inspection (if $V_2 = \mu_1/\varepsilon$). But local hedging, and indeed any committing policy, does worse because it must label item 1 as obligatory-inspection or non-inspection before learning V_2. This construction closely mirrors that of Beyhaghi and Kleinberg (2019, Example 1).

5 Combinatorial Pandora's box problems

We show in this section how local hedging can also be used to provide approximately optimal policies in combinatorial versions of the nonobligatory inspection problem. To this end, we consider the model from Section 2 with two changes.

First, the DM must now select not necessarily just one item, but a set of items satisfying some constraints. Below, we denote by $\mathcal{S} = \{n \in [N] \mid \mathbb{S}_n = 1\}$ the DM's selected set of items. The DM's choice must satisfy certain *feasibility* constraints. We encode the constraints via a set of feasible sets of items, $\mathcal{F} \subseteq 2^{[N]} \setminus \emptyset$, and the DM's choice must be an element of $\mathcal{F}$. Thus, the process of inspecting and selecting items continues until the DM has selected a feasible set of items. We call $\mathcal{F}$ the problem's *constraints*. We assume $\mathcal{F}$ is upward closed, meaning $\mathcal{S}' \supseteq \mathcal{S} \in \mathcal{F}$ implies $\mathcal{S}' \in \mathcal{F}$.

Second, the DM's total cost may now depend not just on the costs paid to inspect and select boxes, but also on the selected set $\mathcal{S}$. In particular, a *terminal cost* function $h : \mathcal{F} \to \mathbb{R}_{\geq 0}$ exists such that the DM's total cost is

$$C = \sum_{n\in[N]} (\mathbb{S}_n V_n + \mathbb{I}_n c_n) + h(\mathcal{S}).$$

We call the pair $(\mathcal{F}, h)$ a *combinatorial model*, as the constraints and terminal cost function together encode all of the combinatorial structure. The model in Section 2, where the DM selects exactly one item, corresponds to $\mathcal{F} = \{\{n\} \mid n \in [N]\}$ and $h(\mathcal{S}) = 0$.

A combinatorial model $(\mathcal{F}, h)$ together with the price distribution G_n and the inspection costs c_n defines an *instance* of a *combinatorial nonobligatory inspection problem*. If we additionally impose the constraint that the DM can only select inspected items, i.e. $\mathbb{S}_n \leq \mathbb{I}_n$, we obtain an instance of a combinatorial *obligatory* inspection problem. We let $\Pi^{\mathrm{NOI}}(\mathcal{F}, h)$ (resp., $\Pi^{\mathrm{OI}}(\mathcal{F}, h)$) denote the set of policies for nonobligatory (resp., obligatory) inspection problems with combinatorial model $(\mathcal{F}, h)$.

Organization of the rest of this section Section 5.1 generalizes our lower bound, Theorem 3.4, to combinatorial selection. Section 5.2 similarly generalizes our upper bound on local hedging, Theorem 4.4, to combinatorial selection. Section 5.3 combines local hedging with the obligatory inspection policies of Singla (2018) to obtain the first approximation algorithms for combinatorial selection under nonobligatory inspection.

5.1 Lower bound on optimal cost in the combinatorial setting

We now give a lower bound on the optimal expected cost for combinatorial selection. Like the single-item selection case, our lower bound is based on the expected value of a one-shot problem using NOI-surrogate prices. In single-item selection, the one-shot problem is simply taking the minimum of the NOI-surrogate prices. In combinatorial selection, the one-shot problem is instead a one-shot version of the optimization problem induced by the combinatorial model, as captured by the following definition.

Definition 5.1 (Surrogate cost)**.** The *optimal* OI-*surrogate cost*, denoted Z^{OI}, is

$$Z^{\mathsf{NOI}} = \min_{\mathcal{S}\in\mathcal{F}} \left(\sum_{n\in\mathcal{S}} W_n^{\mathsf{NOI}} + h(\mathcal{S}) \right). \tag{NOI-SC}$$

Similarly, we define optimal OI-surrogate and LH-surrogate costs, denoted Z^{OI} and Z^{LH}, by replacing W_n^{NOI} with W_n^{OI} and $W_n^{\mathsf{LH}(p_n)}$, respectively.

To understand the connection between the surrogate cost and the surrogate prices in Section 3, consider the single-item selection case, in which the constraint set consists of all the singletons and the terminal cost function is identically 0. In that case, the expression in (NOI-SC) reduces to

$$Z^{\mathsf{NOI}} = \min_{n\in[N]} W_n^{\mathsf{NOI}},$$

which is the cost of the one-shot problem from Theorem 3.4. More generally, (NOI-SC) describes the optimal cost of a one-shot problem when prices are given by W_n^{NOI} and the DM incurs no inspection costs, but does incur a terminal cost when selecting a set of items. In that case, the DM will inspect all items–as learning their prices is free–and then select the minimum cost set.

Like in Section 3, the NOI-surrogate cost provides a lower bound for the combinatorial nonobligatory inspection model.

Theorem 5.2 (Combinatorial selection lower bound)**.** *Consider combinatorial selection with model $(\mathcal{F}, h)$ under nonobligatory inspection. The optimal policy's expected total cost satisfies*

$$\min_{\pi\in\Pi^{\mathsf{NOI}}(\mathcal{F},h)} \mathbf{E}[C^\pi] \geq \mathbf{E}[Z^{\mathsf{NOI}}].$$

We prove Theorem 5.2 in Appendix A. The proof follows the same outline as that of Theorem 3.4, namely finding a suitable submartingale, with some extra complications due to the combinatorial nature of the problem.

Obligatory inspection In light of Proposition 3.6, it is natural to ask whether an analogous result exists for the combinatorial obligatory inspection problem. Singla (2018, Lemma 2.2) shows that the OI-surrogate cost is a lower bound to the optimal cost under obligatory inspection, that is,

$$\min_{\pi \in \Pi^{\mathsf{OI}}} \mathbf{E}[C^\pi] \geq \mathbf{E}[Z^{\mathsf{OI}}]. \tag{5.1}$$

In other words, in the combinatorial model, surrogate costs provide a benchmark against which to compare different policies, but even in the obligatory inspection model, they cease to be a tight benchmark for the optimal cost.

5.2 Local hedging for combinatorial nonobligatory inspection

We show in this section how to extend the local hedging policy from single-item selection to combinatorial nonobligatory inspection. Underlying the logic of local hedging in single-item inspection is knowledge of a policy for obligatory inspection—the so called Weitzman's rule—that applies across all instances of the single-item obligatory inspection. In the combinatorial case, local hedging builds on the existence of such a policy for the combinatorial obligatory inspection problem:

Definition 5.3 (Local hedging in combinatorial nonobligatory inspection)**.** Fix a $\pi \in \Pi^{\mathsf{OI}}(\mathcal{F}, h)$ for combinatorial obligatory inspection and a vector $\mathbf{p} = (p_1, \dots, p_N)$ of hedging probabilities. The π *with local* $\mathbf{p}$*-hedging policy*, denoted $\mathsf{LH}[\pi] \in \Pi^{\mathsf{NOI}}(\mathcal{F}, h)$ (we leave the $\mathbf{p}$ implicit), is the following two-stage policy:

- Using the hedging probabilities $\mathbf{p}$, the DM determines the set of inspection and non-inspection items.
- The DM then runs policy π on the resulting combinatorial obligatory inspection problem.

Our main result below shows, roughly speaking, that if π is a β-approximation for combinatorial selection under obligatory inspection, and if all items admit a local α-approximation, then $\mathsf{LH}[\pi]$ is a $\alpha\beta$-approximation for combinatorial selection under nonobligatory inspection. However, it turns out we need a slightly stronger hypothesis than $\mathsf{LH}[\pi]$ being a β-approximation relative to the optimal policy's expected cost, namely the left-hand side of (5.1). Instead, we need it to be a β-approximation relative to the expected NOI-*surrogate cost*, namely the right-hand side of (5.1). Fortunately, as we discuss in Section 5.3 below, all the results of Singla (2018) yield approximation algorithms relative to this stricter baseline.

Theorem 5.4. *Let $\pi \in \Pi^{\mathsf{OI}}(\mathcal{F}, h)$ be a policy for combinatorial obligatory inspection for a given combinatorial model $(\mathcal{F}, h)$. Suppose that for all hidden price distributions and inspection costs, π satisfies $\mathbf{E}[C^\pi] \leq \beta \mathbf{E}[Z^{\mathsf{OI}}]$. Then for all hidden price distributions and inspection costs, if all items admit a local α-approximation, then using $\mathsf{LH}[\pi]$ with the corresponding hedging probabilities yields expected cost bounded by*

$$\mathbf{E}[C^{\mathsf{LH}[\pi]}] \leq \alpha\beta \mathbf{E}[Z^{\mathsf{NOI}}] \leq \alpha\beta \min_{\pi' \in \Pi^{\mathsf{NOI}}(\mathcal{F}, h)} \mathbf{E}[C^{\pi'}].$$

In particular, $\mathsf{LH}[\pi]$ always yields a $\frac{4}{3}\beta$-approximation for combinatorial selection with nonobligatory inspection.

Table 5.1: Approximation ratios achieved by combining local hedging with policies of Singla (2018) for different obligatory inspection problems

Problem & $\mathcal{F}$	Terminal cost h	Approximation ratio $\frac{4}{3}\beta$
Min-cost matroid basis	0	$\frac{4}{3}$
Min-cost set cover	0	$\min\{O(\log n), \frac{4}{3}f\}$
Min-cost feedback vertex set	0	$O(\log n)$
Facility location ($\mathcal{F} = 2^{[N]} \setminus \{\emptyset\}$)	$\sum_{n\in[N]} \min_{s\in\mathcal{S}} d(n,s)$	2.4814
Steiner tree ($\mathcal{F} = 2^{[N]}$)	Min-Steiner-Tree($[N] \setminus \mathcal{S}$)	4

We defer the proof of Theorem 5.4 to Appendix A, giving a brief outline below. There are two main steps. The first step is to show

$$\mathbf{E}[C^{\mathsf{LH}[\pi]}] \le \beta\mathbf{E}[Z^{\mathsf{LH}}].$$

Fortunately, this follows immediately from (5.1), the assumption that π is a β-approximation relative to the expected OI-surrogate cost under obligatory inspection, and the fact that after local hedging labels items, it transforms the problem into a obligatory inspection problem. This leaves the second step, which is to show

$$\mathbf{E}[Z^{\mathsf{LH}}] \le \alpha\mathbf{E}[Z^{\mathsf{NOI}}],$$

after which Theorem 5.2 completes the proof. The second step generalizes the main task in the proof of Theorem 4.3, which is to show $\mathbf{E}[\min_{n\in[N]} W_n^{\mathsf{LH}(p_n)}] \le \alpha\mathbf{E}[\min_{n\in[N]} W_n^{\mathsf{NOI}}]$. The single-item selection proof proceeds by, roughly speaking, replacing the LH-surrogate prices with α-inflated NOI-surrogate prices one by one. Essentially the same procedure works for combinatorial selection. The main subtlety is that each surrogate price replacement might change the minimizing set of items $\mathcal{S} \in \mathcal{F}$ (see Definition 5.1). The key idea is to express the surrogate cost in terms of a minimum of one item's surrogate price and a quantity that depends only on other items' surrogate prices.

5.3 Extending obligatory inspection results of Singla (2018) to nonobligatory inspection

Theorem 5.4 leaves open the question of whether policies exists for the obligatory combinatorial inspection model that are good approximations of the optimal policy. It turns out that the answer is yes for a number of fundamental problems like matroid basis, set cover, facility location, Steiner-tree, and feedback vertex set. Indeed, Singla (2018, Theorem 1.2) constructs policies satisfying the precondition of Theorem 5.4 for these and other combinatorial models in the *obligatory inspection* setting. Combining these with local hedging yields approximation algorithms for several combinatorial models, as summarized in Table 5.1.

Remark 5.5 (Baseline of Singla's results (Singla, 2018)). While the main result of Singla (2018, Theorem 1.2) is stated as comparing a policy's performance to that of the optimal obligatory inspection policy, namely $\min_{\pi\in\Pi^{\mathsf{OI}}(\mathcal{F},h)} \mathbf{E}[C^\pi]$, inspecting the proof (Singla, 2018, Section 3.2) reveals that all the approximation ratios actually hold relative to the expected OI-surrogate cost $\mathbf{E}[Z^{\mathsf{OI}}]$, as required by our Theorem 5.4.

As Table 5.1 illustrates, Theorem 5.4 allows us to tackle nonobligatory inspection combinatorial models in a variety of settings, which have been largely unexplored because of the difficulties introduced by nonobligatory inspection already in the single-item selection case. Consider for instance the uncapacitated *facility location* problem, in which given a graph with vertices $[N]$ and edges E, the DM must choose a set of locations $\mathcal{S}$ at which to open facilities. The DM wants to minimize the cost of the opened facilities, while at the same time minimizing the distance $d : V \times V \mapsto \mathbb{R}$ of the facilities to those locations at which no facilities are opened.

Another notable family of combinatorial problems is min-cost matroid basis, of which minimum spanning tree is a special case. Thanks to the fact that the optimal algorithm for deterministic min-cost matroid basis is greedy, Singla's results yield an algorithm with $\beta = 1$, so local hedging achieves an approximation ratio of at most $\frac{4}{3}$.

We refer the reader to Singla (2018) for detailed descriptions of the problems in Table 5.1. We emphasize that the power of our approach is not that it addresses any particular combinatorial problem, but rather its *compositionality*. Algorithms and performance guarantees for combinatorial selection under obligatory inspection naturally carry over to local hedging, provided the guarantees are relative to expected OI-surrogate cost.

6 Conclusion and Discussion

In this work, we introduce a new approach to approximately solving Pandora's box problems with nonobligatory inspection. Our approach, *local hedging*, maintains the simplicity and compositionality of the elegant policies available for Pandora's box problems with obligatory inspection. One can view local hedging as a randomized reduction that turns nonobligatory inspection problems into obligatory inspection problems. The result is the first approximation algorithms for combinatorial Pandora's box problems under nonobligatory inspection.

We believe the local hedging technique has potential to be used beyond the setting of this paper. In the rest of this section, we outline the possibilities and obstacles to using local hedging in two additional settings. Section 6.1 discusses *reward maximization* Pandora's box problems, in contrast to the cost minimization setting we focus on. Section 6.2 is *Markovian bandit superprocesses*, which significantly generalize Pandora's box models.

6.1 Reward maximization

In Pandora's box problems in the reward maximization setting, instead of each item having a hidden price, each item has a *hidden reward*. The objective is to maximize expected reward of selected items minus inspection costs (possibly plus a terminal reward in the combinatorial setting).

Nearly all of the core definitions can be translated directly between the cost and reward settings. The rule of thumb is that one can recover definitions for rewards by interpreting them as negative costs, and vice versa. We do this, for instance, when translating Doval's characterization of the one-item subproblem (Doval, 2018, Proposition 0) from rewards to costs. The definitions of local hedging and local approximation can similarly be translated from costs to rewards. More generally, if one allows negative hidden prices or hidden rewards, then the two settings are essentially the same.

With that said, the reward maximization setting is typically studied under the assumption that each item's hidden reward is nonnegative, just as we assume that each item's hidden price is nonnegative. While this does not fundamentally alter any of the definitions, it does have an impact on our main local approximation result, Theorem 4.3. It states that in the cost setting, any item admits a local $\frac{4}{3}$-approximation. However, critical to the proof is the fact that an item's reservation price is bounded. In particular, it is at least the item's inspection cost. But in the rewards setting, an item's *reservation value* (namely its negative reservation price) is not similarly bounded. For nonnegative hidden rewards, an item's reservation value can have *arbitrarily large* ratio relative to the mean hidden reward and inspection cost.

The impact is that if one attempts to translate Theorem 4.3 to the rewards setting, most of the proof translates straightforwardly, but at the very end, one finds that the worst possible approximation ratio is only $\frac{1}{2}$. This is a disappointing result, because Beyhaghi and Kleinberg (2019) point out that the following trivial randomized committing policy is a $\frac{1}{2}$-approximation in the rewards case: with equal probability, commit to either never inspecting any items or never selecting any items without inspection.[6] So while local hedging may still yield a superior result for items that admit local α-approximations for $\alpha > \frac{1}{2}$, it appears that simple local hedging alone cannot replicate the best simple approximation algorithms for the rewards setting, which achieve a $\frac{4}{5}$-approximation (Guha et al., 2008).

It is an interesting open question whether the definition of local approximation can be altered in a way that enables improved guarantees in the rewards setting. One idea would be to combine the multiplicative suboptimality factor currently considered with an additive suboptimality gap, which we suspect could rule out the worst-case examples that admit only local $\frac{1}{2}$-approximation.

6.2 Bandit superprocesses

As Doval (2018) demonstrates through multiple examples, the core reason why Pandora's box problems are harder under nonobligatory inspection than obligatory inspection is that nonobligatory inspection gives the DM two possible actions to take on each box. This core difficulty also manifests in a significantly more general model, that of *Markovian bandit superprocesses*. We see potential for applying local hedging to certain cases of this more general setting.

We begin with some background. Roughly speaking, a *Markovian bandit process* is a type of Markov decision process in which the DM must at each time step choose between advancing one of multiple independent Markov chains (Gittins et al., 2011). The traditional setting in which bandit processes are studied is infinite-horizon discounted reward problems (Weber, 1992), but variants exist for unconstrained-but-finite undiscounted problems, like Pandora's box problems. One example is the model of Dumitriu et al. (2003), which, aside from the assumption of discrete state spaces, is a multistage generalization of single-item selection in the cost minimization setting.

Many definitions and results available for Pandora's box problems under obligatory inspection translate to bandit processes. Most importantly, reservation prices (or reservation values) are a special case of *Gittins indices*, which form the basis of optimal index policies for many varieties of bandit process (Gittins et al., 2011). Gittins indices share the simplicity and compositionality of reservation prices. For instance, Gupta et al. (2019) generalize the combinatorial selection results of Singla (2018) to a more general model resembling that of Dumitriu et al. (2003). The core reason why these results work is that even though the DM has many choices of Markov chains to advance at each time step, only one action, namely "advance", is possible within each Markov chain.

[6] Curiously, we are not aware of a similar trivial randomized algorithm for the costs setting.

Markovian bandit *superprocesses* generalize bandit processes by replacing the independent Markov chains with independent Markov *decision processes*. That is, now each of the independent processes may present the DM with multiple possible actions. In Pandora's box problems, uninspected items under nonobligatory inspection are an example of this, as they allow either inspection or immediate selection. While one can define Gittins indices for bandit superprocesses, unlike the simple bandit process case, they generally do not yield optimal policies. The only known exception is when each Markov decision process satisfies a condition known as *Whittle's condition* (Whittle, 1980; Glazebrook, 1982). Roughly speaking, a Markov decision process satisfies Whittle's condition if one would be willing to commit to a state-to-action mapping ahead of time, then always use that to determine which action to play, regardless of the states of other Markov decision processes in the bandit superprocess.

Our notion of local α-approximation can be viewed as a *novel relaxation of Whittle's condition*. Specifically, in our Pandora's box setting, satisfying Whittle's condition amounts to admitting a local 1-approximation. One can easily generalize the definition of local approximation to the more general superprocess setting, although there are some subtleties to work out about what types of randomness should be allowed in the state-to-action mapping. But we suspect that analogues of our Theorems 4.4 and 5.4, which show that all items admitting local α-approximations yields an α-approximation for the overall problem, should hold for general superprocesses, with essentially the same proof outline.

The significance of local approximations relaxing Whittle's condition is that the set of Markov decision processes that admit local α-approximations for values of α reasonably close to 1 is likely to be much richer than those that satisfy Whittle's condition. For instance, in Pandora's box with nonobligatory inspection, the only items that satisfy Whittle's condition are those for which inspection is never worthwhile. In contrast, all items admit local $\frac{4}{3}$-approximations. We there believe that local hedging and local approximations give a promising new angle of attack for deriving approximation algorithms for bandit superprocesses.

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A Deferred proofs

Lemma 3.3 (Surrogate prices solve one-item nonobligatory inspection)**.** *For all $r \in \mathbb{R}$,*

$$\bar{C}^{\mathrm{OI}}_{\mathrm{item}}(r) = \min\{c + \mathbf{E}[\min\{V, r\}, r]\} = \mathbf{E}[\min\{W^{\mathrm{OI}}, r\}],$$
$$\bar{C}^{\mathrm{NOI}}_{\mathrm{item}}(r) = \min\{c + \mathbf{E}[\min\{V, r\}, r, \mu]\} = \mathbf{E}[\min\{W^{\mathrm{NOI}}, r\}].$$

Proof. The obligatory inspection statement is standard (see, e.g., Kleinberg et al., 2016, Lemma 1), so we turn immediately to the nonobligatory inspection statement. We consider two cases:

Case 1: $u^{\mathrm{rsv}} < u^{\mathrm{bkp}}$. Expanding both sides using Definition 3.2, we aim to show

$$\mathbf{E}\big[\min\{\max\{V, u^{\mathrm{rsv}}\}, u^{\mathrm{bkp}}, r\}\big] = \min\big\{\mathbf{E}\big[\min\{\max\{V, u^{\mathrm{rsv}}\}, r\}\big], \mu\big\}. \tag{A.1}$$

We clearly have

$$\mathbf{E}\big[\min\{\max\{V, u^{\mathrm{rsv}}\}, u^{\mathrm{bkp}}, r\}\big] \le \mathbf{E}\big[\min\{\max\{V, u^{\mathrm{rsv}}\}, r\}\big], \tag{A.2}$$

and using Definition 3.2 and the $u^{\mathrm{rsv}} < u^{\mathrm{bkp}}$ assumption, we compute

$$\begin{aligned}\mathbf{E}\big[\min\{\max\{V, u^{\mathrm{rsv}}\}, u^{\mathrm{bkp}}, r\}\big] &\le \mathbf{E}\big[\min\{\max\{V, u^{\mathrm{rsv}}\}, u^{\mathrm{bkp}}\}\big] \\ &= \mu_n + \mathbf{E}[(u^{\mathrm{rsv}} - V)^+] - \mathbf{E}[(V - u^{\mathrm{bkp}})^+] \\ &= \mu + c - c = \mu.\end{aligned} \tag{A.3}$$

One of (A.2) or (A.3) holds with equality (because $r \le u^{\mathrm{bkp}}$ or $r \ge u^{\mathrm{bkp}}$), implying (A.1).

Case 2: $u^{\mathrm{rsv}} \ge u^{\mathrm{bkp}}$. Expanding both sides using Definition 3.2, we aim to show

$$\min\{\mu, r\} = \min\big\{\mu, \mathbf{E}[\min\{W^{\mathrm{OI}}, r\}]\big\}.$$

The left-hand side is greater than or equal to the right-hand side, so it remains only to show the reverse inequality. By Definition 3.2, we have $u^{\mathrm{rsv}} \le W^{\mathrm{OI}}$ with probability 1, so it suffices to show $\mu \le u^{\mathrm{rsv}}$. This holds due to the $u^{\mathrm{rsv}} \ge u^{\mathrm{bkp}}$ assumption, (RP) and (BP), and the following computation:

$$\begin{aligned}\mu = \mathbf{E}[V] &= u^{\mathrm{rsv}} + \mathbf{E}[(V - u^{\mathrm{rsv}})^+] - \mathbf{E}[(u^{\mathrm{rsv}} - V)^+] \\ &\le u^{\mathrm{rsv}} + \mathbf{E}[(V - u^{\mathrm{bkp}})^+] - \mathbf{E}[(u^{\mathrm{rsv}} - V)^+] \\ &= u^{\mathrm{rsv}} + c - c = u^{\mathrm{rsv}}.\end{aligned}$$

□

Theorem 3.4 (Single-item selection lower bound)**.** *In nonobligatory inspection single-item selection, the optimal policy's expected total cost satisfies*

$$\bar{C}^{\mathrm{NOI}} = \min_{\pi \in \Pi^{\mathrm{NOI}}} \mathbf{E}[C^{\pi}] \ge \mathbf{E}\left[\min_{n \in [N]} W^{\mathrm{NOI}}_n\right]. \tag{LB-OPT}$$

Proof. Consider an arbitrary policy π for the DM. After t rounds, the policy has inspected some items and possibly selected one. Let[7]

$$C(t) = \text{total inspection and selection cost paid by } \pi \text{ during } \{0, \dots, t-1\},$$

$$W_n(t) = \begin{cases} 0 & \text{if any item is selected by } \pi \text{ during } \{0, \dots, t-1\} \\ V_n & \text{if } n \text{ is inspected, but not selected, by } \pi \text{ during } \{0, \dots, t-1\} \\ W_n^{\mathsf{NOI}} & \text{otherwise,} \end{cases}$$

$$\mathcal{I}(t) = \text{information } \pi \text{ gains from inspections during } \{0, \dots, t-1\},$$

$$K(t) = C(t) + \mathbf{E}\left[\min_{n \in [N]} W_n(t) \;\middle|\; \mathcal{I}(t)\right].$$

After $N+1$ rounds, the DM will have selected an item and the process will have terminated, so $K(N+1) = C^\pi$. We also have $K(0) = \mathbf{E}[\min_{n \in [N]} W^{\mathsf{NOI}}]$, so it suffices to show that $\{K(t)\}_t$ is a submartingale with respect to $\{\mathcal{I}(t)\}_t$.

Below, to reduce clutter, we abbreviate $\mathbf{E}[\cdot \mid \mathcal{I}(t)]$ to $\mathbf{E}_t[\cdot]$.

We aim to show $K(t) \le \mathbf{E}_t[K(t+1)]$. To do so, we consider each action the DM might take on each item $m \in [N]$. For each action, we write the difference $\mathbf{E}_t[K(t+1)] - K(t)$ in terms of the quantity

$$R_{\neq m}(t) = \min_{n \in [N] \setminus \{m\}} W_n(t).$$

The fact that $R_{\neq m}(t)$ depends only on $W_n(t)$ for $n \neq m$ implies the following observations.

(O1) If the DM takes an action on item m at time t, then $R_{\neq m}(t) = R_{\neq m}(t+1)$.
(O2) $R_{\neq m}(t)$ is conditionally independent of $W_m(t)$ given $\mathcal{I}(t)$.

With the appropriate notation and the above observations in hand, we can show the difference $\mathbf{E}_t[K(t+1)] - K(t)$ is nonnegative no matter which action the DM takes.

- If item m is closed and the DM inspects it, then

$$\begin{aligned} \mathbf{E}_t[K(t+1)] - K(t) &= -\mathbf{E}_t[\min\{W_n^{\mathsf{NOI}}, R_{\neq m}(t)\}] + c_n + \mathbf{E}_t[\min\{V_n, R_{\neq m}(t+1)\}] \\ &= -\mathbf{E}_t[\min\{W_n^{\mathsf{NOI}}, R_{\neq m}(t)\}] + c_n + \mathbf{E}_t[\min\{V_n, R_{\neq m}(t)\}]. \end{aligned}$$

  The second equality follows by (O1), and its right-hand side is nonnegative by (O2) and Lemma 3.3.
- If item m is closed and the DM selects it (without inspection), then

$$\mathbf{E}_t[K(t+1)] - K(t) = -\mathbf{E}_t[\min\{W_n^{\mathsf{NOI}}, R_{\neq m}(t+1)\}] + \mu_n \quad = -\mathbf{E}_t[\min\{W_n^{\mathsf{NOI}}, R_{\neq m}(t)\}] + \mu_n.$$

  Again, the second equality follows by (O1), and its right-hand side is nonnegative by (O2) and Lemma 3.3.
- If item m is open and the DM selects it, then

$$\mathbf{E}_t[K(t+1)] - K(t) = -\mathbf{E}_t[\min\{V_n, R_{\neq m}(t)\}] + V_n.$$

  This is nonnegative because V_n is known to the DM at time t, so $V_n = \mathbf{E}_t[V_n]$. □

[7] Below, we allow t to be greater than the number of rounds that π takes to select an item. We follow the convention that the process simply remains static after an item is selected.

Theorem 5.4. *Let $\pi \in \Pi^{\mathsf{OI}}(\mathcal{F}, h)$ be a policy for combinatorial obligatory inspection for a given combinatorial model $(\mathcal{F}, h)$. Suppose that for all hidden price distributions and inspection costs, π satisfies $\mathbf{E}[C^\pi] \leq \beta \mathbf{E}[Z^{\mathsf{OI}}]$. Then for all hidden price distributions and inspection costs, if all items admit a local α-approximation, then using $\mathsf{LH}[\pi]$ with the corresponding hedging probabilities yields expected cost bounded by*

$$\mathbf{E}[C^{\mathsf{LH}[\pi]}] \leq \alpha\beta\mathbf{E}[Z^{\mathsf{NOI}}] \leq \alpha\beta \min_{\pi' \in \Pi^{\mathsf{NOI}}(\mathcal{F},h)} \mathbf{E}[C^{\pi'}].$$

In particular, $\mathsf{LH}[\pi]$ always yields a $\frac{4}{3}\beta$-approximation for combinatorial selection with nonobligatory inspection.

Proof. We actually show the following stronger result: if for each item, a local hedging probability exists giving a local α-approximation (or better), then using those same local hedging probabilities makes $\mathsf{LH}[\pi]$ a $\alpha\beta$-approximation. This implies the theorem because by Theorem 4.3, such local hedging probabilities exist for some $\alpha < 4/3$.

Because the second stage of $\mathsf{LH}[\pi]$ runs π, the fact that $\mathbf{E}[C^\pi] \leq \beta\mathbf{E}[Z^{\mathsf{OI}}]$ implies

$$\mathbf{E}[C^{\mathsf{LH}[\pi]}] \leq \beta\mathbf{E}[Z^{\mathsf{LH}}].$$

Combining this observation with Theorem 5.2, it suffices to show that for appropriately chosen local hedging probabilities, we have

$$\mathbf{E}[Z^{\mathsf{LH}}] \leq \alpha\mathbf{E}[Z^{\mathsf{NOI}}] \tag{A.4}$$

Before proving (A.4) formally, let us outline the main idea, which is essentially a generalization of the proof of Theorem 4.4. Starting from Z^{LH} and replacing each item's LH-surrogate price with α times its NOI-surrogate price one by one, resulting in αZ^{NOI}. Because local hedging gives a local α-approximation (Definition 4.2) for each item, each replacement only increases the expected value.

To formalize the above outline, we need notation for describing the replacement of surrogate prices one by one. To that end, let

$$W_n^{(m)} = \begin{cases} \alpha W_n^{\mathsf{NOI}} & \text{if } n \leq m \\ W_n^{\mathsf{LH}(p_n)} & \text{if } n > m, \end{cases}$$

$$Z^{(m)} = \min_{\mathcal{S} \in \mathcal{F}} \left(\sum_{n \in \mathcal{S}} W_n^{(m)} + h(\mathcal{S}) \right).$$

Then it suffices to show $\mathbf{E}[Z^{(m-1)}] \leq \mathbf{E}[Z^{(m)}]$, as then

$$\mathbf{E}[Z^{\mathsf{LH}}] = \mathbf{E}[Z^{(0)}] \leq \cdots \leq \mathbf{E}[Z^{(N)}] = \alpha\mathbf{E}[Z^{\mathsf{NOI}}],$$

where the equalities at either end follow from Definition 5.1.

To show $\mathbf{E}[Z^{(m-1)}] \leq \mathbf{E}[Z^{(m)}]$, the key idea is to split the minimization over $[N] \in \mathcal{F}$ into cases based on whether item m is in the optimizing set. We therefore define

$$X_{\neq m} = \min_{\mathcal{S} \in \mathcal{F} \,|\, m \notin \mathcal{S}} \left(\sum_{n \in \mathcal{S}} W_n^{(m)} + h(\mathcal{S}) \right),$$

$$Y_{\neq m} = \min_{\mathcal{S} \in \mathcal{F} \,|\, m \in \mathcal{S}} \left(\sum_{n \in \mathcal{S} \setminus \{m\}} W_n^{(m)} + h(\mathcal{S}) \right).$$

Because $W_n^{(m-1)} = W_n^{(m)}$ for all $n \neq m$, we have

$$\begin{aligned} Z^{(m-1)} &= \min\{X_{\neq m}, Y_{\neq m} + W_m^{\mathsf{LH}(p)}\}, \\ Z^{(m)} &= \min\{X_{\neq m}, Y_{\neq m} + \alpha W_m^{\mathsf{NOI}}\}. \end{aligned}$$

Note also that $X_{\neq m}$ and $Y_{\neq m}$ are independent of $W_m^{\mathsf{LH}(p)}$ and W_m^{NOI}, as they depend on the surrogate prices of all items *except* item m. Using this and the fact that local hedging gives a local α-approximation for item m (Definition 4.2), we compute

$$\begin{aligned} \mathbf{E}[Z^{(m-1)} \mid X_{\neq m}, Y_{\neq m}] &= Y_{\neq m} + \mathbf{E}[\min\{W_m^{\mathsf{LH}(p)}, X_{\neq m} - Y_{\neq m}\} \mid X_{\neq m}, Y_{\neq m}] \\ &\leq Y_{\neq m} + \mathbf{E}[\min\{\alpha W_m^{\mathsf{NOI}}, X_{\neq m} - Y_{\neq m}\} \mid X_{\neq m}, Y_{\neq m}] \\ &= \mathbf{E}[Z^{(m)} \mid X_{\neq m}, Y_{\neq m}], \end{aligned}$$

which implies $\mathbf{E}[Z^{(m-1)}] \leq \mathbf{E}[Z^{(m)}]$, as desired. ◻

Theorem 5.2 (Combinatorial selection lower bound)**.** *Consider combinatorial selection with model $(\mathcal{F}, h)$ under nonobligatory inspection. The optimal policy's expected total cost satisfies*

$$\min_{\pi \in \Pi^{\mathsf{NOI}}(\mathcal{F},h)} \mathbf{E}[C^\pi] \geq \mathbf{E}[Z^{\mathsf{NOI}}].$$

Proof. This proof follows essentially the same steps as the proof of Theorem 3.4 above, but with some extra complications to handle the combinatorial aspect.

Consider an arbitrary policy π for the DM. After t rounds, the policy has inspected some items and possibly selected one. Let[8]

$$\begin{aligned} C(t) &= \text{total inspection and selection cost paid by } \pi \text{ during } \{0, \dots, t-1\}, \\ W_n(t) &= \begin{cases} 0 & \text{if any item is selected by } \pi \text{ during } \{0, \dots, t-1\} \\ V_n & \text{if } n \text{ is inspected, but not selected, by } \pi \text{ during } \{0, \dots, t-1\} \\ W_n^{\mathsf{NOI}} & \text{otherwise,} \end{cases} \\ \mathcal{I}(t) &= \text{information } \pi \text{ gains from inspections during } \{0, \dots, t-1\}, \\ \mathcal{F}(t) &= \left\{\mathcal{S} \in \mathcal{F} \;\middle|\; \mathcal{S} \text{ contains all items selected by } \pi \text{ during } \{0, \dots, t-1\}\right\} \\ Z(t) &= \min_{\mathcal{S} \in \mathcal{F}(t)} \left(\sum_{n \in \mathcal{S}} W_n(t) + h(\mathcal{S})\right), \\ K(t) &= C(t) + \mathbf{E}[Z(t) \mid \mathcal{I}(t)]. \end{aligned}$$

After $N+1$ rounds, the DM will have selected an item and the process will have terminated, so $K(N+1) = C^\pi$. We also have $K(0) = \mathbf{E}[\min_{n \in [N]} W^{\mathsf{NOI}}]$, so it suffices to show that $\{K(t)\}_t$ is a submartingale with respect to $\{\mathcal{I}(t)\}_t$.

Below, to reduce clutter, we abbreviate $\mathbf{E}[\cdot \mid \mathcal{I}(t)]$ to $\mathbf{E}_t[\cdot]$.

[8]Below, we allow t to be greater than the number of rounds that π takes to select an admissible set of items. We follow the convention that the process simply remains static after such a set is selected.

We aim to show $K(t) \le \mathbf{E}_t[K(t+1)]$. To do so, we consider each action the DM might take on each item $m \in [N]$. For each action, write the difference $\mathbf{E}_t[K(t+1)] - K(t)$ in terms of the quantities

$$\begin{aligned} X_{\neq m}(t) &= \min_{\mathcal{S} \in \mathcal{F}(t) \mid m \notin \mathcal{S}} \left(\sum_{n \in \mathcal{S}} W_n(t) + h(\mathcal{S}) \right), \\ Y_{\neq m}(t) &= \min_{\mathcal{S} \in \mathcal{F}(t) \mid m \in \mathcal{S}} \left(\sum_{n \in \mathcal{S} \setminus \{m\}} W_n(t) + h(\mathcal{S}) \right), \\ R_{\neq m}(t) &= X_{\neq m}(t) - Y_{\neq m}(t). \end{aligned}$$

We use the convention that a minimum over an empty set is ∞. That is, if all sets in $\mathcal{F}(t)$ contain m, then $X_{\neq m}(t) = \infty$, and similarly if none contain m, then $Y_{\neq m}(t) = \infty$.[9]

These quantities above give a useful decomposition of $Z(t)$, namely that for any item m,

$$\begin{aligned} Z(t) &= \min\{X_{\neq m}(t), Y_{\neq m}(t) + W_m(t)\} \\ &= Y_{\neq m}(t) + \min\{W_m(t), R_{\neq m}(t)\}. \end{aligned}$$

The fact that $Y_{\neq m}(t)$ and $R_{\neq m}(t)$ depend only on $W_n(t)$ for $n \neq m$ implies the following observations.

(O1) If the DM takes an action on item m at time t, then $X_{\neq m}(t) \le X_{\neq m}(t+1)$ and $Y_{\neq m}(t) = Y_{\neq m}(t+1)$, so $R_{\neq m}(t) \le R_{\neq m}(t+1)$.
(O2) $R_{\neq m}(t)$ is conditionally independent of $W_m(t)$ given $\mathcal{I}(t)$.

With the appropriate notation and the above observations in hand, we can show the difference $\mathbf{E}_t[K(t+1)] - K(t)$ is nonnegative no matter which action the DM takes.

- If item m is closed and the DM inspects it, then

$$\begin{aligned} \mathbf{E}_t[K(t+1)] - K(t) &= -\mathbf{E}_t[Y_{\neq m}(t) + \min\{W_n^{\mathsf{NOI}}, R_{\neq m}(t)\}] \\ &\quad + c_n + \mathbf{E}_t[Y_{\neq m}(t+1) + \min\{V_n, R_{\neq m}(t+1)\}] \\ &\ge -\mathbf{E}_t[\min\{W_n^{\mathsf{NOI}}, R_{\neq m}(t)\}] + c_n + \mathbf{E}_t[\min\{V_n, R_{\neq m}(t)\}]. \end{aligned}$$

  The inequality follows by (O1), and its right-hand side is nonnegative by (O2) and Lemma 3.3.
- If item m is closed and the DM selects it (without inspection), then

$$\begin{aligned} \mathbf{E}_t[K(t+1)] - K(t) &= -\mathbf{E}_t[Y_{\neq m}(t) + \min\{W_n^{\mathsf{NOI}}, R_{\neq m}(t)\}] \\ &\quad + \mathbf{E}_t[Y_{\neq m}(t+1) + \min\{\mu_n, \infty\}] \\ &= -\mathbf{E}_t[\min\{W_n^{\mathsf{NOI}}, R_{\neq m}(t)\}] + \mu_n. \end{aligned}$$

  The second equality follows by (O1), and its right-hand side is nonnegative by (O2) and Lemma 3.3. The infinity appears because $X_{\neq m}(t+1) = \infty$ due to m being selected at time t.

[9]Provided the policy π never fails to select a feasible set of items, it will never be the case that both $X_{\neq m}(t)$ and $Y_{\neq m}(t)$ are infinite.

- If item m is open and the DM selects it, then

$$\begin{aligned}\mathbf{E}_t[K(t+1)] - K(t) &= -\mathbf{E}_t[Y_{\neq m}(t) + \min\{W_n^{\mathrm{NOI}}, R_{\neq m}(t)\}] \\ &\quad + \mathbf{E}_t[Y_{\neq m}(t+1) + \min\{V_n, \infty\}] \\ &= -\mathbf{E}_t[\min\{W_n^{\mathrm{NOI}}, R_{\neq m}(t)\}] + V_n.\end{aligned}$$

  The second equality follows by (O1), and its right-hand side is nonnegative because V_n is known to the DM at time t, so $V_n = \mathbf{E}_t[V_n]$. Again, the infinity appears because $X_{\neq m}(t+1) = \infty$ due to m being selected at time t. □

B Alternative proof of the lower bound for single-item selection

In this section, we give an alternative proof of Theorem 3.4 by reducing it to a result of Beyhaghi and Kleinberg (2019), which is a tighter lower bound on $\inf_\pi \mathbf{E}[C^\pi]$. The reason we do not directly use Beyhaghi and Kleinberg's result for our proof is that, as will soon become clear, their result is less explicit than Theorem 3.4. Rather than a single expression that works for all policies π, their result bounds $\mathbf{E}[C^\pi]$ using an expression which itself depends on π. But it is instructive to present their result and show how it implies our weaker but more explicit bound.

We note that although Beyhaghi and Kleinberg (2019) consider the reward-maximization setting instead of the cost-minimization setting, the definitions and proofs we consider in this section easily translate to the cost setting. We believe Beyhaghi and Kleinberg's approach could also be used to obtain a policy-dependent version of Theorem 5.2, but this is less immediate.

The main idea behind the approach Beyhaghi and Kleinberg (2019) take is this: to prove a policy-dependent bound, one should use a *policy-dependent analogue of surrogate prices*. One way of doing so is the following.

Definition B.1. Let π be a policy for the nonobligatory inspection problem. The *π-surrogate price* of item n is the random variable

$$W_n = \begin{cases} W_n^{\mathrm{OI}} & \text{if } \pi \text{ inspects item } n \\ \mu & \text{if } \pi \text{ does not inspect item } n. \end{cases}$$

Unlike the other types of surrogate prices we consider, two different items' π-surrogate prices are *not necessarily independent*. This is because the hidden price revealed by inspecting one item may influence the decision of whether to inspect another.[10] Despite this subtlety, Beyhaghi and Kleinberg (2019) use π-surrogate prices to prove the following bound.

Proposition B.2 (cost analogue of Beyhaghi and Kleinberg (2019, Lemma 16))**.** *In the nonobligatory inspection problem, the expected total of policy π satisfies*

$$\mathbf{E}[C^\pi] \geq \mathbf{E}\left[\min_{n \in [N]} W_n\right].$$

As we show below, this result implies Theorem 3.4.

[10]This is true even under local hedging, which uses independent randomness to commit to inspect-before-select or no-inspect for each item. The issue is that local hedging might not inspect an item even if it is inspect-before-select, and this decision depends on the hidden prices of other items.

Alternative proof of Theorem 3.4. In light of Proposition B.2, it suffices to show

$$\mathbf{E}\left[\min_{n\in[N]} W_n\right] \geq \mathbf{E}\left[\min_{n\in[N]} W_n^{\mathrm{NOI}}\right]. \tag{B.1}$$

Similarly to the proof of Theorem 3.4, we show this by replacing π-surrogate prices with NOI-surrogate prices one-by-one. Specifically, it suffices to show that for each $n \in [N]$, we have[11]

$$\mathbf{E}[\min\{\dots, W_{n-1}, W_n^{\pi}, W_{n+1}^{\mathrm{NOI}}, \dots\}] \geq \mathbf{E}[\min\{\dots, W_{n-1}^{\pi}, W_n^{\mathrm{NOI}}, W_{n+1}^{\mathrm{NOI}}, \dots\}],$$

as chaining these inequalities together for all $n \in [N]$ yields (B.1). The above holds if for all $r \in \mathbb{R}$,

$$\mathbf{E}[\min\{W_n^{\pi}, r\}] \geq \mathbf{E}[\min\{W_n^{\mathrm{NOI}}, r\}].$$

By Definition B.1 and Lemma 3.3, we have, as desired,

$$\begin{aligned}\mathbf{E}[\min\{W_n^{\pi}, r\}] &\geq \min\big\{\mathbf{E}[\min\{W_n^{\mathrm{OI}}, r\}], \min\{\mu_n, r\}\big\} \\ &= \min\big\{\mathbf{E}[\min\{W_n^{\mathrm{OI}}, r\}], \mu_n\big\} \\ &= \mathbf{E}[\min\{W_n^{\mathrm{NOI}}, r\}].\end{aligned}$$

□

[11]The expression below implicitly assumes $3 \leq n \leq N-2$ for simplicity of notation, but we both require and prove the analogous inequality for the edge cases.