# Combinatorial Selection with Costly Information

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#### Abstract

We consider a class of optimization problems over stochastic variables where the algorithm can learn information about the value of any variable through a series of costly steps; we model this information acquisition process as a Markov Decision Process (MDP). The algorithm's goal is to minimize the cost of its solution plus the cost of information acquisition, or alternately, maximize the value of its solution minus the cost of information acquisition. Such bandit superprocesses have been studied previously but solutions are known only for fairly restrictive special cases.

We develop a framework for approximate optimization of bandit superprocesses that applies to arbitrary processes with a matroid (and in some cases, more general) feasibility constraint. Our framework establishes a bound on the optimal cost through a novel cost amortization; it then couples this bound with a notion of local approximation that allows approximate solutions for each component MDP in the superprocess to be composed without loss into a global approximation.

We use this framework to obtain approximately optimal solutions for several variants of bandit superprocesses for both maximization and minimization. We obtain new approximations for combinatorial versions of the previously studied Pandora's Box with Optional Inspection and Pandora's Box with Partial Inspection; as well as approximation algorithms for a new problem that we call the Weighing Scale problem.

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# Contents



## <span id="page-2-0"></span>1 Introduction

Consider a biotechnology company investing in multiple drug discovery projects. Each individual project can involve a multitude of decisions with uncertain outcomes. Typically very few projects will make it to trials and FDA approval at which point they can begin to yield rewards. The company needs to determine how to distribute resources across these projects without knowing upfront which one will succeed and to what extent. Many decision making problems call for upfront investment in information acquisition (such as how much reward a project will generate) before optimizing over multiple options (such as which project to seek FDA approval for). Some other examples include an oil company's search for a good drilling site; a construction company exploring different design ideas for a project; a manufacturer performing market research before determining which products to make and in what quantity, etc.

As a toy example of costly information acquisition in the context of optimization, consider what we call the Weighing Scale problem. The algorithm is presented with n alternatives with unknown costs,  $X_1, \dots, X_n$ , of which it needs to pick a cheap one. Each cost  $X_i$  is distributed independently according to a known distribution. The only way the algorithm can learn any further information about the  $X_i$ 's is to use a weighing scale at the additional cost of 1 per use: for any index i and target t, the weighing scale returns  $\mathbb{1}[X_i \leq t]$ . What information, if any, should the algorithm gather before making its selection so as to minimize the expected cost of the chosen alternative plus the number of weighings?

In this paper we study a class of problems that we call Costly Information Combinatorial Selection (henceforth, CICS), of which Weighing Scale is an example. Our goal is to pick a subset of n alternatives with stochastic values satisfying a feasibility constraint. We can obtain information about the value of any individual alternative through a sequence of costly actions tailored to that alternative. In the minimization version of the problem our objective is to minimize the total value of the chosen alternatives plus the cost of information acquisition. In the maximization version we maximize the total value of the chosen subset minus the cost of information acquisition.

The algorithmic challenge of CICS stems from two levels of decision-making that influence each other – which alternative to explore in each step; as well as how to explore the chosen alternative (e.g. which target values  $t$  to use in the Weighing Scale setting). A policy can, in general, make these choices adaptively based on the information obtained on previous steps.

A classic example of the CICS is Weitzman's Pandora's Box problem. Here each alternative is hidden inside a closed box; the goal is to select a single alternative; the algorithm can observe the value of the alternative by opening, a.k.a. "inspecting", each box at a cost; and any alternative must be inspected before being selected. Pandora's Box admits an astonishingly simple optimal algorithm: each box is assigned an index based on its cost and value distribution; for minimization, boxes are inspected in increasing<sup>[1](#page-2-1)</sup> order of index until the algorithm finds a value that is smaller than all remaining indices and accepts that value. The indices do not depend on the algorithm's stochastic trajectory; the order of information acquisition is therefore non-adaptive. Essentially, the indices capture all of the salient information about each individual box that can then be used by the algorithm to determine the order in which to explore the boxes.

Interestingly, this optimality structure extends to a large class of feasibility constraints over Pandora's Boxes: [Singla](#page-35-0) [\[2017\]](#page-35-0) showed that Weitzman's indices and the corresponding greedy ordering algorithm achieve approximate optimality for essentially all selection problems that admit greedy-style (a.k.a. frugal) approximation algorithms.

However, the landscape changes if the algorithm has multiple modes of information acquisition available for each alternative. For example, if inspection becomes optional, that is, if the algorithm is allowed to select an alternative without observing its value, the optimal algorithm is no longer non-adaptive and the problem becomes NP-hard. Much effort has been devoted recently towards developing approximations for variants of Pandora's Box, including the optional inspection version and the so-called partial inspection version. See, for example, the survey by [Beyhaghi and Cai](#page-34-1) [\[2024\]](#page-34-1). These results are generally tailored to the specifics of the variants they study and do not extend to other models of information acquisition. Moreover, they do not always generalize to combinatorial settings.

<span id="page-2-1"></span><sup>1</sup>For the maximization version of the problem, boxes are opened in decreasing order of index.

A natural question is whether index-based policies can be extended to other CICS settings: can we distill the essence of the information acquisition process for each alternative into a Weitzman-style index, and build an approximation algorithm based on it? This is the question we seek to answer in this paper.

We develop a framework for designing approximately-optimal policies for both the minimization and maximization versions of CICS. Our framework has two components. First we develop a bound on the cost of the optimal solution based on a novel cost amortization technique that generalizes the Gittins index to Markov Decision Processes. This bound allows us to disentagle the exploration of different alternatives into individual "local" problems. We then show that a certain notion of **local approximation** applied to each individual exploration problem can be combined via an index-based policy to obtain a global approximation guarantee. This approach is a strengthening and generalization of the "Local Hedging" technique developed by [Scully and Doval](#page-35-1) [\[2024\]](#page-35-1) in the context of Pandora's Box with optional inspection.

We instantiate our framework with new approximation results for previously studied variants of Pandora's Box as well as the new weighing scale problem described above. Our framework and results extend seamlessly to combinatorial selection problems over matroid constraints as well as to combinations of different variants of information acquisition. For example, when given a selection problem with some weighing-scale-alternatives and some optional-inspection-alternatives, our approach can effortlessly combine local approximations for each variant to obtain a combined guarantee for the whole problem.

#### Bandit superprocesses and our cost amortization framework

Selection problems with costly information are closely related to a class of sequential decision-making problems called bandit superprocesses, which generalize Markovian multi-armed bandits (MAB) [\[Gittins et al.,](#page-34-2) [2011\]](#page-34-2). In an MAB, the algorithm is given  $n$  Markovian "arms"; at every step the algorithm can pull any one of the arms and receive a reward; the state of the pulled arm then evolves according to the underlying Markov process. [Gittins](#page-34-3) [\[1979\]](#page-34-3) showed that an index-based policy is optimal for this problem. MAB is usually studied in an infinite horizon setting with discounted or averaged reward. [Dumitriu et al.](#page-34-4) [\[2003\]](#page-34-4) extended this approach to the finite horizon setting (where Markov chains have terminal states) and [Gupta et al.](#page-35-2) [\[2019\]](#page-35-2) further extended it to combinatorial selection. Weitzman's index-policy for Pandora's Box is a special case of these results.

In a general bandit superprocess, the  $n$  arms are Markov Decision Processes (MDPs) comprising multiple possible actions with different costs/rewards. While it is possible to treat the superprocess as a single large MDP, the exponential size of this MDP makes this approach computationally infeasible, and it is desirable to consider algorithms that "solve" each component process separately and combine the solutions together. [Whittle](#page-35-3) [\[1980\]](#page-35-3) showed that certain bandit superprocesses admit simple index-policies just like the MAB special case. In particular, when each MDP in the superprocess admits an unambiguously optimal static policy, these policies can be composed to obtain a globally optimal policy through appropriately defined indices. However, in the absence of such a strong local optimality condition, the globally optimal policy may take actions that are suboptimal for the MDP they're taken in.

Much like Whittle's work, our approach considers a local problem  $(M, y)$  for each MDP M where the algorithm is presented with a deterministic outside option y and at any step can either choose to take an action in  $M$  or accept the outside option  $y$ . The optimal solution to this local problem generates an optimality curve  $f_{\mathcal{M}}(y)$  as a function of the outside option. A primary technical contribution of our work is to convert this optimality curve into a mapping from terminal states of the MDP to "surrogate" costs (Section [3\)](#page-9-0). The surrogate costs essentially allow an algorithm to amortize the costs of actions and pay (a part of) them only when the MDP terminates. We develop a recursive "water filling" algorithm for cost amortization that ensures that good, i.e. low cost or high reward, terminal states are responsible for paying most of the cost share. The overall index of the MDP is then defined to be the cost of the best-case scenario – the lowest possible surrogate cost that is instantiated. In the special case of Markov chains, our water filling amortization exactly recovers the Gittins index.

A key property of the surrogate costs we define is their independence from the actions chosen by the algorithm in the MDP (Lemma [3.3](#page-11-1) in Section [3\)](#page-9-0). The corresponding indices can therefore be defined and computed independently of the algorithm's stochastic trajectory, much like in Gittins or Weitzman's setting. This allows us to compose these costs and obtain a global bound on the value of the optimal solution. However, the independence comes at a loss – the surrogate costs provide a bound that is not exact for general MDPs (in contrast to Whittle's results, for example).

Our approach is heavily influenced by the work of [Kleinberg et al.](#page-35-4) [\[2016\]](#page-35-4) who presented an amortization viewpoint for Weitzman's index for classical Pandora's Box, as well as for its extension to Markov chains. The amortization viewpoint is also implicit in the work of [Singla](#page-35-0) [\[2017\]](#page-35-0), who bounds the optimal cost by considering a phantom "free information" world. Our amortized cost shares reflect a similar argument. We emphasize that while bounds on the optimal solution's value have been established previously for the single-selection CICS in special cases, a general bound was not known previously. Our bound applies to any feasibility constraint over any bandit superprocess where the constituent MDPs have a DAG-like structure.

#### Local and semilocal approximation

Next we consider approximations to the CICS through a class of algorithms called "commitment policies". A commitment policy for an instance of the CICS specifies a (random) action for the algorithm to take at any state of any component MDP.[2](#page-4-0) The commitment for each alternative depends on its own state and not on the states of other processes. This turns each MDP into a (suboptimal) Markov chain. Armed with these commitments, the algorithm faces a combinatorial MAB which, based on the prior discussion, we know how to solve.

Given the characterization of the optimal cost in terms of properties of the local games  $(M, y)$  defined above, one reasonable approach is to try to find a commitment policy for each MDP that is approximately optimal for the local game. However, this does not work. Consider, for example, a minimization MDP  $\mathcal M$ with two actions. The first action leads to a deterministic value of 1, and the second leads to a stochastic value of 0 or 1 with probability  $1/2$  each. Then, the first action provides a 2-approximation for the local game  $(M, y)$ : its expected cost is min $(y, 1)$  whereas the optimal cost of the local game is half of that quantity, achieved by taking the stochastic option. Now consider a single-selection superprocess with  $n$  copies of the above MDP. If we commit to the deterministic action in each MDP, our cost is deterministically 1, whereas taking the stochastic action in each costs us  $1/2<sup>n</sup>$  – an exponential gap!

[Scully and Doval](#page-35-1) [\[2024\]](#page-35-1) describe how to fix this problem in the context of Pandora's Box with optional inspection and define a local approximation as follows. Let  $f_{\mathcal{M}}(\cdot)$  and  $f_{\pi}(\cdot)$  denote the optimality curves of the MDP M and the commitment policy  $\pi$  for M, respectively, in the local game  $(M, y)$ . The local approximation factor for minimization achieved by  $\pi$  for the MDP M is  $\alpha \geq 1$  if we have

<span id="page-4-2"></span>
$$
f_{\pi}(\alpha y) \leq \alpha f_{\mathcal{M}}(y) \,\forall y \in \mathbb{R}.\tag{1}
$$

For maximization, we likewise require  $f_{\pi}(\frac{y}{\alpha}) \geq \frac{1}{\alpha} f_{\mathcal{M}}(y)$  for all y. It then holds that any commitment policy that achieves an  $\alpha$ -local approximation for each constituent MDP achieves a global  $\alpha$ -approximation.<sup>[3](#page-4-1)</sup> We show in Section [4](#page-15-0) that this composition result holds for general CICS.

Informally speaking, local approximation is equivalent to establishing second order stochastic dominance between the surrogate costs of the MDP and the committing policy. We also explore the stronger condition of first order stochastic dominance between surrogate costs, that we call pointwise approximation, which can be easier to establish in some settings. These characterizations allows us to design commitment policies that achieve a good approximation.

In some settings, the local approximation condition can be challenging to satisfy. For example, if  $f<sub>M</sub>$ is near-linear, then Equation [\(1\)](#page-4-2) is essentially equivalent to requiring  $f_{\mathcal{M}} = f_{\pi}$ .<sup>[4](#page-4-3)</sup> We therefore explore a weakening of the condition by allowing for additive deviations from  $(1)$ . We call this property Semilocal Approximation, and show that it leads to improved approximation bounds for some settings (Section [7.3\)](#page-28-0).

<span id="page-4-0"></span><sup>&</sup>lt;sup>2</sup>E.g., in the weighing scale problem, for each  $X_i$ , we may specify targets  $t_1^i, t_{11}^i, t_{12}^i$ , etc. such that  $X_i$  is first compared against  $t_1^i$ , and then against  $t_{11}^i$  or  $t_{12}^i$  depending on whether th

<span id="page-4-1"></span> $3$ Observe that Equation [\(1\)](#page-4-2) is stronger than simply requiring the commitment policy to be approximately optimal for the local game  $(M, y)$  at every possible outside option y (in other words,  $f_{\pi}(y) \leq \alpha f_{\mathcal{M}}(y)$ ).

<span id="page-4-3"></span><sup>&</sup>lt;sup>4</sup>That is,  $\pi$  must be unambiguously optimal for M; and no suboptimal policy can provide any local approximation.

#### Our results

We establish new or improved approximation factors for three variants of the CICS. In each case our results apply to matroid feasibility constraints in the given setting. For the first and second settings, our results extend also to feasibility constraints admitting frugal approximation algorithms as defined in [Singla](#page-35-0) [\[2017\]](#page-35-0).

- Pandora's Box with Partial Inspection (Section [5\)](#page-17-0). This setting is an extension of the classical PB where in addition to opening the box, the algorithm can choose to "peek" into the box at a cheaper cost and learn its value; boxes must still be opened before being selected. We consider the minimization cost and learn its value; boxes must still be opened before being selected. We consider the minimization<br>version of this problem and present a  $\sqrt{2}$ -approximation using the notion of local approximation. The minimization version of this problem has not been studied previously.
- Weighing Scale Problem (Section [6\)](#page-21-0). This is the problem described in the introduction, and is a new problem introduced in this paper. We obtain a logarithmic approximation in key parameters of the problem using the notion of pointwise approximation.
- Pandora's Box with Optional Inspection (Section [7\)](#page-25-0). This setting is an extension of the classical PB where the algorithm is allowed to select boxes without opening them. We consider the maximization version of this problem. We show that no policy can achieve better than a 0.5-local approximation. On the other hand, semilocal approximation allows us to achieve a 0.582-approximation. The maximization version of the problem is extensively studied but previous results only apply to the single selection setting. Our result, in contrast, extends to matroid feasibility constraints.

Our cost amortization framework is presented in Section [3](#page-9-0) and local approximation in Section [4.](#page-15-0)

#### <span id="page-5-0"></span>1.1 Related work

Bandit superprocesses (a.k.a. BSPs) were first defined by [Nash](#page-35-5) [\[1973\]](#page-35-5). [Gittins](#page-34-3) [\[1979\]](#page-34-3) first showed that optimal policies for the MAB are indexable and introduced the Gittins index (see, also, [Nash](#page-35-6) [\[1980\]](#page-35-6), [Weber](#page-35-7) [\[1992\]](#page-35-7)). [Weitzman](#page-35-8) [\[1979\]](#page-35-8) independently proved the optimality of index policies for the Pandora's Box special case. Further progress on combinatorial selection over Markov chains was made by [Dumitriu et al.](#page-34-4) [\[2003\]](#page-34-4), [Singla](#page-35-0) [\[2017\]](#page-35-0) and [Gupta et al.](#page-35-2) [\[2019\]](#page-35-2), with the last paper showing the approximate optimality of indexing-based frugal algorithms for this setting. For general BSPs, [Whittle](#page-35-3) [\[1980\]](#page-35-3) provided an alternate proof of the Gittins index theorem and introduced optimality curves and the Whittle integral, extending the optimality of indexed policies to BSPs with dominant local policies (known in the literature as "Whittle's condition"). [Glazebrook](#page-35-9) [\[1982\]](#page-35-9) showed that BSPs are not indexable in general. Several decades later [Brown and Smith](#page-34-5) [\[2013\]](#page-34-5) proved that the Whittle integral provides an upper bound on the optimal value for a general maximization BSP; their work applies to the single selection problem in the infinite horizon discounted setting and is based on dynamic programming. Our amortization-based bound is identical to the Whittle integral but is algorithmic and constructive; applies to the finite horizon setting as well as to combinatorial selection; and, as a side product, provides surrogate costs that prove useful in designing approximations.

There is little work on BSPs that do not satisy Whittle's condition, outside of the variants of Pandora's Box. The only such work we are aware of is by [Ke and Villas-Boas](#page-35-10) [\[2019\]](#page-35-10), who look at a problem with two specific symmetric MDPs and an arbitrary outside option, and develop an adaptive (but complicated) exact algorithm for this problem. We refer the reader to the survey by [Hadfield-Menell and Russell](#page-35-11) [\[2015\]](#page-35-11) for further discussion of BSPs.

The idea of augmenting the inspection process of a Pandora's box in order to explore more interesting decision-making settings has been a very active line of research over the last years. Most literature on the optional inspection variant of the problem is for single-item selection, as opposed to combinatorial variants. Study of it was initiated by [Guha et al.](#page-35-12) [\[2008\]](#page-35-12), who give a 4/5-approximation for the maximization setting; and [Doval](#page-34-6) [\[2018\]](#page-34-6), who characterized the solution to the single-box problem and proved certain conditions under which the Gittins policy remains optimal for single-item selection in the maximization setting (though the results naturally extend to the minimization setting). Beyond this results are separated by whether they are for the minimization or maximization setting. For the minimization setting, [Scully and](#page-35-1) [Doval](#page-35-1) [\[2024\]](#page-35-1) proved a composition theorem for a special case of local approximation (Definition [7\)](#page-15-1) and used it to construct a committing policy with a 4/3-approximation guarantee. This result extends as-is to the combinatorial setting via [Singla](#page-35-0) [\[2017\]](#page-35-0)'s frugal algorithms framework.

In the maximization setting, [Beyhaghi and Kleinberg](#page-34-7) [\[2019\]](#page-34-7) and [Guha et al.](#page-35-12) [\[2008\]](#page-35-12) give approximation guarantees for committing policies. Furthermore, [Fu et al.](#page-34-8) [\[2022\]](#page-34-8) and [Beyhaghi and Cai](#page-34-9) [\[2022\]](#page-34-9) introduced polynomial time approximation schemes that for any  $\varepsilon > 0$  compute a policy that is at least a  $(1 - \varepsilon)$ approximation. However, all of these results are for the single-item selection setting and do not extend to combinatorial settings such as matroid selection. Prior to this work, no policy was known to obtain better than a straightforward 1/2-approximation for matroid selection PBOI. We show in Section [7](#page-25-0) that while local approximation is inadequate for beating the  $1/2$ , our novel approach of semilocal approximation provides a 0.582 approximation.

The partial inspection variant of Pandora's Box has primarily been studied in the context of maximiza-tion. [Aouad et al.](#page-34-10) [\[2020\]](#page-34-10) provide a  $(1 - 1/e)$ -approximation via a committing policy, and show that, in fact, any committing algorithm or its negation (flipping which box should be partially opened versus fully opened) admits a  $(1/2)$ -approximation to the optimal utility. We note that this already highlights a significant difference between the minimization and maximization settings for this variant. Whereas for maximization one can essentially flip a coin to decide which of the two actions to commit to, obtaining the optimal's utility for the committed action and non-negative utility for the action that was not selected, the same approach cannot be applied for minimization as the cost suffered by a bad flip could result to arbitrarily bad approximations. [Beyhaghi](#page-34-11) [\[2019\]](#page-34-11) introduces a more general inspection model, where the searcher has k different methods for inspecting each box, and can select at most one of them. They provide a  $(1 - 1/e)$ -approximation that applies to k-element selection, but is computationally inefficient when  $k$  is large. To our knowledge, partial inspection has not been studied in the context of minimization.

Other extensions of Pandora's Box include settings with combinatorial rewards [\[Olszewski and Weber,](#page-35-13) [2015\]](#page-35-13); combinatorial costs [\[Berger et al., 2023\]](#page-34-12); correlated values [\[Chawla et al., 2019,](#page-34-13) [2021,](#page-34-14) [Gergatsouli and](#page-34-15) [Tzamos, 2024\]](#page-34-15); or constraints on the order of inspection [\[Esfandiari et al., 2019,](#page-34-16) [Boodaghians et al., 2020,](#page-34-17) [Bowers and Waggoner, 2024\]](#page-34-18). We refer the reader to the survey by [Beyhaghi and Cai](#page-34-1) [\[2024\]](#page-34-1) for further discussion of Pandora's Box.

The weighing scale problem has not been studied previously, although [Hoefer et al.](#page-35-14) [\[2024\]](#page-35-14) study a similar setting, where each alternative can be weighed against the *median* of its distribution (conditioned on any past weighings). They consider a model where the algorithm is provided a budget on the number of weighings and wants to find the best alternative subject to the budget; and design a constant factor approximation in this setting.

### <span id="page-7-0"></span>2 Preliminaries

We begin by defining a single component of our algorithmic problem that involves learning information about a random variable through a series of costly steps. We represent this process as a Markov Decision Process (henceforth, MDP), instantiated over this random variable. An MDP is described by specifying its state space, action set and how states transition upon taking actions. In the case of Costly Information MDPs, states represent the information the algorithm has gained about the corresponding random variable. Accordingly, we associate each state with the conditional value distribution it represents. Formally:

**Definition 1.** A Costly Information MDP for a random variable X is a tuple  $\mathcal{M}_X = (S, \sigma, A, c, \mathcal{D}, V, T)$ , where S is a set of states,  $\sigma \in S$  is the starting state,  $A(\cdot)$  maps states to sets of actions,  $c(\cdot)$  is a cost function mapping actions to costs, and D is a transition matrix. For each pair of states  $s, s' \in S$  and each action  $a \in A(s), \mathcal{D}(s, a, s') \in [0, 1]$  specifies the probability of transitioning to s' upon taking action a in state s; naturally,  $\sum_{s'} \mathcal{D}(s, a, s') = 1$  for all states  $s \in S$  and actions  $a \in A(s)$ .

Furthermore,  $V(\cdot)$  is a function mapping states to distributions over values.  $V(\sigma)$  is the (unconditional) distribution of X and  $V(s)$  is the posterior distribution of X conditioned on being in state  $s \in S$ . As such, V satisfies the rules of conditional probability: for all states  $s \in S$  and all actions  $a \in A(s)$ , we have  $V(s) = \sum_{s'} \mathcal{D}(s, a, s')V(s')$ . We also write  $v(s) := E[V(s)]$ . Finally,  $T \subseteq S$  is the set of terminal states. Terminal states have only one action available, called the "accept" action. This accept action comes at no cost; results in a value of  $v(s)$  at terminal state  $s \in T$ ; and terminates the MDP.

We call the process  $\mathcal{M}_X$  a Costly Information Markov chain if there is only one action (accept or advance) available at every state  $s \in S$ . When clear from the context, we will drop the subscript X. For simplicity, we will often refer to these components simply as "MDPs" and "Markov chains".

At the expense of blowing up the size of the state space and action set, we will assume that the sets of actions  $A(s)$  and  $A(s')$  are disjoint for  $s \neq s'$ , and furthermore that each state in the MDP  $\mathcal{M}_X$  is reached via a unique sequence of actions starting from  $\sigma$ ; in other words, the state "memorizes" the sequence of actions taken to reach it and thus our MDPs have a tree-like structure. We also assume that each MDP  $\mathcal{M}_X$  has a finite horizon: there is a constant  $H$  such that any state reached via a sequence of  $H$  steps is a terminal state. Finally, for simpler exposition, we will assume that the state spaces, action sets and the supports of the random variables X are finite. We note that our entire framework seamlessly extends to continuous settings.

We are now ready to define selection problems with costly information:

Definition 2. The Costly Information Combinatorial Selection problem (henceforth, CICS) is defined over a ground set of n random variables,  $X_1, X_2, \cdots, X_n$ ; a feasibility constraint  $\mathcal{F} \subseteq 2^{[n]}$ ; and a costly information MDP  $\mathcal{M}_i := \mathcal{M}_{X_i} = (S_i, \sigma_i, A_i, c_i, \mathcal{D}_i, V_i, T_i)$  for each variable  $X_i$ . The constraint  $\mathcal F$  corresponds to an upwards closed set in the minimization version (henceforth, min-CICS), and to a downwards closed set in the maximization version (henceforth, max-CICS).

An algorithm (a.k.a. adaptive policy) for CICS proceeds as follows. Let  $S \subseteq [n]$  denote the set of indices of all terminated MDPs. Let  $s_i$  denote the current state of the MDP  $\mathcal{M}_i$  at any point of time during the process. Initially  $S = \emptyset$  and  $s_i = \sigma_i$  for all  $i \in [n]$ . The algorithm chooses at every step an index  $i \in [n] \setminus S$ corresponding to a non-terminated MDP  $\mathcal{M}_i$  and an action  $a_i \in A_i(s_i)$ . It then follows the action at a cost of  $c_i(a_i)$ . If  $a_i$  is the accept action (i.e.  $s_i \in T_i$  is a terminal state), it adds i to S and collects the value  $v_i(s_i)$ . Otherwise, it updates the state of  $\mathcal{M}_i$  to a new state drawn from the distribution  $\mathcal{D}(s_i, a_i, \cdot)$ , while the states of all other MDPs  $\mathcal{M}_{i'}$  with  $i' \neq i$  remain unchanged. Observe that the algorithm can make both of these choices – the index of the MDP to move in and the action to take in that MDP – adaptively depending on the evolution of all  $n$  MDPs in previous steps.

• For min-CICS, the algorithm terminates as soon as  $S \in \mathcal{F}$ . The objective of min-CICS is to find an algorithm with minimum total cost, defined as the expectation (over the randomness of the algorithm and the underlying processes) of the total cost of all actions undertaken by the algorithm until termination plus the values accrued from accept actions.

• For max-CICS, the algorithm needs to ensure feasibility by selecting at every step indices  $i \in [n] \setminus S$ such that  $S \cup \{i\} \in \mathcal{F}$ . The objective of max-CICS is to find an algorithm with maximum utility, defined as the expectation (over the randomness of the algorithm and the underlying processes) of the total value accrued from accept actions minus the total cost of all actions undertaken by the algorithm until termination.

Feasibility Constraints. We focus primarily on matroid feasibility constraints: each MDP corresponds to an element of some known matroid  $\mathbb{M} = ([n], \mathcal{I})$ . For min-CICS, F is the collection of all sets that contain a basis of M; for max-CICS,  $\mathcal F$  contains all the independent sets of M. Observe that single-element selection, where the algorithm's goal is to accept one MDP (e.g. in the classic version of Pandora's Box or the weighing scale problem described in the introduction), is a special case of matroid selection.

Committing Algorithms. A committing policy  $\pi$  for an MDP  $\mathcal{M} = (S, \sigma, A, c, \mathcal{D}, V, T)$  maps every state  $s \in S$  to a distribution  $\pi(s)$  over actions in  $A(s)$ . An MDP M coupled with a committing policy  $\pi$  defines a Costly Information Markov chain, denoted  $\mathcal{M}^{\pi}$ , where the single action available in every state is given by the distribution over actions specified by  $\pi$ . Observe that the states in  $\mathcal{M}^{\pi}$  are a subset of the states in M and terminal states in M continue to be terminal states in  $\mathcal{M}^{\pi}$ . We use  $\mathcal{C}(\mathcal{M})$  to denote the set of all committing policies  $\pi$  for an MDP  $\mathcal{M}$ .

A commitment for a CICS is defined as a tuple  $\Pi = (\pi_1, \dots, \pi_n)$  of committing policies  $\pi_i \in \mathcal{C}(\mathcal{M}_i)$ , one for each of the n MDPs comprising the CICS. A committing algorithm under  $\Pi$  chooses a feasible index i at every step, samples an action from the distribution  $\pi_i(s_i)$  and plays that action. Observe that although the action taken by the algorithm is pre-chosen in every state, the algorithm can choose the index of the MDP to move in at every step adaptively based on the evolution of all  $n$  MDPs in previous steps. Committing algorithms are therefore adaptive algorithms, although they are a simpler and smaller class relative to the class of all algorithms.

Henceforth, we will focus primarily on matroid-min-CICS. We detail the changes needed to the framework for the maximization setting in Appendix [E.](#page-49-0) Following the work of [Singla](#page-35-0) [\[2017\]](#page-35-0), our results will apply to any combinatorial setting for which the underlying constraint admits an efficient frugal approximation algorithm. We describe this extension in Appendix [F.](#page-52-0)

### <span id="page-9-0"></span>3 An Amortization Framework

In this section, we develop a novel cost amortization technique that will allow us to lower bound the cost of the optimal adaptive algorithm for matroid-min-CICS. We first establish our amortization for the special case of Markov chains (Section [3.1\)](#page-9-1). We then connect our amortization framework to the notion of optimality curves (Section [3.2\)](#page-10-0) and use this connection to extend our approach to general MDPs (Section [3.3\)](#page-11-0). Omitted proofs are presented in Appendix [A.](#page-36-0)

#### <span id="page-9-1"></span>3.1 Amortized Surrogate Costs for Markov Chains

Let  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  be an instance of matroid-min-CICS where every MDP  $\mathcal{M}_i$  is a Markov chain. In this setting, an algorithm chooses an index  $i \in [n]$  at every step and advances  $\mathcal{M}_i$  along the unique action available. It is well known that the optimal algorithm is an index policy: we associate every state  $s_i \in S_i$ for  $i \in [n]$  with an index, and advance the chain with the minimum index at every step. Importantly, the index is independent of the evolution of states in the algorithm, and depends solely on the description of the Markov chain. The policy can therefore be computed easily. We will now describe a new, simple proof of the optimality of index-based policies based on an amortization of action costs.

We begin with some notation. For a Markov chain  $\mathcal{M} = (S, \sigma, A, c, \mathcal{D}, V, T)$  and a state  $s \in S$ , we use  $c(s)$ to denote the cost of the unique action in  $A(s)$ . We use  $T(s) \subseteq T$  to denote the set of terminal states reachable from s. Consider an algorithm that starts in  $\sigma$  and advances the chain M along the unique available action until it reaches a terminal state. For  $s \in S$ , let  $p(s)$  denote the probability that the state s is encountered during this process. Observe that by definition  $p(s) = \sum_{t \in T(s)} p(t)$ ;  $p(\sigma) = 1$ ; and p defines a probability distribution over the terminal states in T. We are now ready to define our notion of cost amortization.

<span id="page-9-3"></span>**Definition 3.** A cost amortization of a Markov chain  $M = (S, \sigma, A, c, D, V, T)$  is a non-negative vector  $b = \{b_{st}\}_{s \in S, t \in T(s)}$  with the property that  $\sum_{t \in T(s)} p(t)b_{st} = p(s)c(s)$  for all states  $s \in S$ . Based on this amortization, we define:

- The amortized cost of a terminal state  $t \in T$  as  $\rho_b(t) := v(t) + \sum_{s:t \in T(s)} b_{st}$ .
- The surrogate cost of the Markov chain M as the random variable  $\rho_{M,b}$  that takes on value  $\rho_b(t)$ for  $t \in T$  with probability  $p(t)$ .
- The index of a state  $s \in S$  of the Markov chain M as  $I_{\mathcal{M},b}(s) = \min_{t \in T(s)} \rho_b(t)$ .

Observe that only the terminal states carry amortized costs; essentially, the amortization distributes the cost of every action across its downstream terminals. The surrogate cost of a Markov chain is the amortized cost of the random terminal state realized by running the chain until it terminates.

Borrowing terminology from [Singla](#page-35-0) [\[2017\]](#page-35-0), we can relate the performance of any algorithm on  $\mathbb{I}$  to its performance in a "free information world" where the algorithm does not pay action costs but, whenever it accepts at a terminal state, is responsible for paying the entire amortized cost at that state. Since the algorithm does not accept every Markov chain it advances, we immediately get the following lower bound.

<span id="page-9-2"></span>**Lemma 3.1.** Consider a matroid-min-CICS  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  over Markov chains and let  $b_i$  be any cost amortization of  $\mathcal{M}_i$  with surrogate cost  $\rho_i := \rho_{\mathcal{M}_i,b_i}$  for all  $i \in [n]$ . Then, the expected cost of any algorithm for  $\mathbb{I}$  is at least  $E\left[\min_{S \in \mathcal{F}} \sum_{i \in S} \rho_i\right]$ .

<span id="page-9-4"></span>We will now exhibit a specific cost amortization and a corresponding algorithm that achieves the lower bound of Lemma [3.1,](#page-9-2) proving optimality. The water filling cost amortization is described algorithmically in a bottom-up fashion. Recall that each state in the Markov chain is associated with a unique sequence of actions that lead to it; let the length of this sequence denote the "level" of the state. We start from the terminal states, and define their total cost share to be equal to their value  $v(t)$ . We then proceed over non-terminal states in decreasing order of level. Each non-terminal state  $s \in S$  distributes its total cost  $c(s)$ across its downstream terminals  $T(s)$ , starting from the terminals with the lowest current total cost, until the equation  $\sum_{t \in T(s)} p(t) b_{st} = p(s) c(s)$  is satisfied. We use  $W^*_{\mathcal{M}}$  to denote the water filling surrogate cost of a Markov chain  $\mathcal{M}$ , and  $I^*_{\mathcal{M}}(s)$  to denote the water filling index of a state s in M.

Example 1. Consider an MDP M whose starting state is given a choice between following one of two Markov chains  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . The Markov chain  $\mathcal{M}_1$  has a single action of cost  $c_1 = 1$ , leading to a terminal of value  $v(t_{11}) = (2/3)$  with probability  $p_{11} = (3/4)$  and to a terminal of value  $v(t_{12}) = 4$  with probability  $p_{12} = (1/4)$ . The Markov chain  $\mathcal{M}_2$  has a single action of cost  $c_2 = (1/8)$ , leading to a terminal of value  $v(t_{21}) = (1/2)$  with probability  $p_{12} = (1/4)$  and to a terminal of value  $v(t_{22}) = 3$  with probability  $p_{22} = (3/4)$ . We pictorially represent  $M$  as well as the water filling amortizations for  $M_1$  and  $M_2$  in Figure [1.](#page-10-1)

<span id="page-10-1"></span>

Figure 1: The water filling amortization for Markov chains  $\mathcal{M}_1$  (left) and  $\mathcal{M}_2$  (right). For each chain, the water filling amortization computes a unique index g (here,  $g_1 = 2$  and  $g_2 = 1$ ), corresponding to the value for which the highlighted area equals the cost of the amortized action. For each terminal t, the corresponding cost share is given by  $(g - v(t))^+$ ; here,  $b_{11} = 4/3$ ,  $b_{21} = 1/2$  and  $b_{12} = b_{22} = 0$ . Therefore, the surrogate cost  $W^*_{\mathcal{M}_1}$  is 2 with probability (3/4) and 4 with probability (1/4). Likewise,  $W^*_{\mathcal{M}_2}$  is 1 with probability (1/4) and 3 with probability (3/4).

We now describe an index-based policy for matroid-min-CICS based on the water filling amortization:

**Definition 4.** The Water Filling Index policy for an instance  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  of matroid-min-CICS chooses at every step the Markov chain  $i^* = \operatorname{argmin}_{i \in \mathcal{F}_S} I_i^*(s_i)$ , where  $s_i$  is the current state of Markov chain  $\mathcal{M}_i$ ; S is the set of terminated (selected) Markov chains; and  $\mathcal{F}_S = \{i : \text{rank}(S \cup \{i\}) > \text{rank}(S)\}.$ 

The water filling amortization ensures that if  $b_{st} > 0$  for some state  $s \in S$  and terminal  $t \in T(s)$ , then the index of all states in the unique path from s to t will be the same. Since the water filling index policy always advances the Markov chain of minimum index, this implies that for any action taken by the policy, any downstream terminal that "owes" a non-zero cost share to that action will be accepted if instantiated. Therefore all the amortized cost shares are paid in expectation, establishing the following optimality result.

<span id="page-10-2"></span>**Theorem 3.2.** For any matroid-min-CICS instance  $\mathbb{I} = (\mathcal{M}_1, \cdots, \mathcal{M}_n, \mathcal{F})$  over Markov chains, the expected cost of the water filling index policy is equal to  $E\left[\min_{S \in \mathcal{F}} \sum_{i \in S} W_{\mathcal{M}_i}^*\right]$ . The policy is therefore optimal for instance I.

#### <span id="page-10-0"></span>3.2 Local Games and Optimality Curves

We now connect the water filling amortization described above to the notion of *optimality curves* for MDPs defined in the work of [Whittle](#page-35-3) [\[1980\]](#page-35-3) and its follow ups. We first define a "local game" for an MDP M.

**Definition 5.** The local game  $(M, y)$  is a single-selection min-CICS with two MDPs, one of which is the MDP  $M$ . The second MDP, a.k.a. the outside option, is a deterministic option with a single state of value y and a single "accept" action available. A policy for the local game advances  $\mathcal M$  for some (zero or non-zero) number of steps and either accepts the deterministic option y or the value from M. Let  $f_{\mathcal{M}}(y)$ denote the expected cost of the optimal policy for the local game  $(\mathcal{M}, y)$ . We refer to the function  $f_{\mathcal{M}}$  as the Optimality Curve of the MDP M.

The surrogate cost of the outside option  $y$  (under any cost amortization) is simply  $y$ ; as a consequence of Theorem [3.2,](#page-10-2) we obtain the following characterization:

**Fact 1.** For any Markov chain M and any  $y \in \mathbb{R}$ , it holds that  $f_{\mathcal{M}}(y) = E \left[\min(y, W_{\mathcal{M}}^*)\right]$ .

Observe from this characterization that the CDF of the surrogate cost  $W^*_{\mathcal{M}}$  can be derived<sup>[5](#page-11-2)</sup> from the optimality curve as  $1 - \frac{d}{dy} f_{\mathcal{M}}(y)$ . In his seminal work, Whittle used optimality curves and dynamic programming to prove that the cost of the optimal policy for the single-selection min-CICS over Markov chains is precisely  $E[\min_i Z_i]$ , where for each  $i \in [n]$ ,  $Z_i$  is a random variable instantiated from a distribution with CDF  $1-\frac{d}{dy}f_{\mathcal{M}}(y)$ . Whittle's proof extends to MDPs where the optimal algorithm for the local game  $(\mathcal{M}, y)$ employs a fixed committing policy  $\pi \in \mathcal{C}(\mathcal{M})$  regardless of the outside option y:  $\pi$  is unconditionally optional. Our water filling approach provides the same bound and optimality result through an alternate, arguably simpler, argument. Moreover, using this equivalence between water filling surrogate costs and optimality curves in the case of Markov chains, we can extend the definition of water filling surrogate costs to arbitrary MDPs:

<span id="page-11-3"></span>**Definition 6.** Let M be an MDP with optimality curve  $f_{\mathcal{M}}$ . The Water Filling Surrogate Cost for M is the random variable  $W^*_{\mathcal{M}}$  generated by picking a value from the CDF  $1 - \frac{d}{dy} f_{\mathcal{M}}(y)$ . That is,  $W^*_{\mathcal{M}}$  is the random variable satisfying  $f_{\mathcal{M}}(y) = E[\min(y, W^*_{\mathcal{M}})]$  for all  $y \in \mathbb{R}$ .

Unfortunately, this definition does not give us insight into how the water filling surrogate costs relate to the structure of the MDP, or whether and how we can extend Lemma [3.1](#page-9-2) or Theorem [3.2](#page-10-2) to arbitrary MDPs. We will now address these issues by directly relating optimality curves to the water filling procedure.

#### <span id="page-11-0"></span>3.3 Water Filling and Surrogate Costs for General MDPs

In the previous sections, we established that in the case of Markov chains, the water filling surrogate cost of a randomly sampled terminal state recovers the optimality curve. The challenge to extending this argument for general MDPs is that each sequence of actions creates a different distribution over the terminal states. We address this with the following lemma:

<span id="page-11-1"></span>**Lemma 3.3.** For any MDP  $\mathcal{M} = (S, \sigma, A, c, \mathcal{D}, V, T)$ , there exists an amortized cost function  $\rho : T \mapsto \Delta(\mathbb{R})$ mapping terminal states to distributions over costs and a non-negative cost sharing vector  $b = \{b_{st}\}_{s \in S, t \in T(s)}$ , such that for all committing policies  $\pi \in \mathcal{C}(\mathcal{M})$ , generating a Markov chain  $\mathcal{M}^{\pi}$  with states  $S_{\pi} \subseteq S$ , terminal states  $T_{\pi} \subseteq T \cap S_{\pi}$ , and a distribution  $p_{\pi}$  over them, the following properties hold:

- 1. Action Independence. The water filling surrogate cost  $W^*_{\mathcal{M}}$  corresponds to sampling a terminal state  $t \sim p_{\pi}$  and then sampling from distribution  $\rho(t)$ .
- 2. Cost Sharing. For all  $t \in T_{\pi}$ , it holds that  $E[\rho(t)] = v(t) + \sum_{s \in S_{\pi}: t \in T_{\pi}(s)} b_{st}$ .
- 3. Cost Dominance. For all  $s \in S_\pi$ , it holds that  $\sum_{t \in T_\pi(s)} p_\pi(t) b_{st} \leq p_\pi(s) c(a)$  for the unique action  $a \in A(s)$  chosen by  $\pi$ .

Lemma [3.3](#page-11-1) allows us to express the water filling surrogate cost of the MDP through a cost sharing and amortization much in the same way as Definition [3](#page-9-3) does for Markov chains. Each terminal is assigned a random amortized cost that is independent of the trajectory of the algorithm. Yet, as the action independence condition states, every committing policy generates the same distribution over amortized costs, matching the water filling surrogate costs defined based on optimality curves. This allows us to account for the eventual cost of any algorithm without worrying about the specific actions it takes.

The lemma also defines action independent cost shares paid by terminals to the upstream actions. However, the action independence comes at a cost. In contrast to the cost shares defined for Markov chains in Definition [3,](#page-9-3) these cost shares only recover a part of the cost of the action. As a result, water filling surrogate costs provide us with a lower bound, but not an exact accounting, of the cost of any algorithm for the CICS. The proof of the following lower bound closely mirrors the lower bound we previously established for Markov chains (Lemma [3.1\)](#page-9-2).

<span id="page-11-4"></span><span id="page-11-2"></span><sup>&</sup>lt;sup>5</sup>In particular, let H and h denote the CDF and PDF of  $W^*_{\mathcal{M}}$  respectively, then we have  $f_{\mathcal{M}}(y) = y(1 - H(y)) + \int_0^y zh(z)dz$ , from which the statement follows by differentiating both sides.



Figure 2: We consider the same setting as in Example [1.](#page-9-4) The optimality curve for MDP  $\mathcal M$  is obtained by the minimum of the optimality curves for  $\mathcal{M}_1$  (orange line) and  $\mathcal{M}_2$  (blue line). From the optimality curve, we see that  $W^*_{\mathcal{M}}$  is 1 with probability  $(1/4)$ , 2.5 with probability  $(1/2)$  and 4 with probability  $(1/4)$ . From Lemma [3.3,](#page-11-1)  $W^*_{\mathcal{M}}$  can be obtained by first sampling from either  $W^*_{\mathcal{M}_1}$  or  $W^*_{\mathcal{M}_2}$  and then applying a randomized mapping  $\rho_{\mathcal{M}}(\cdot)$  to the sampled value. The mapping probabilities are shown in the table.

**Theorem 3.4.** For any matroid-min-CICS instance  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$ , the expected cost of the optimal adaptive policy is at least  $E\left[\min_{S \in \mathcal{F}} \sum_{i \in S} W_{\mathcal{M}_i}^*\right]$ .

We conclude this section with the proof of Lemma [3.3.](#page-11-1)

**Proof of Lemma [3.3.](#page-11-1)** Fix any MDP  $\mathcal{M} = (S, \sigma, A, c, \mathcal{D}, V, T)$ . The proof follows by induction on the horizon H of the MDP. If  $H = 1$ , then the starting state  $\sigma$  is the unique terminal state of M and thus all three conditions are satisfied by the trivial mapping  $\rho(\sigma) := v(\sigma)$ . We now assume that the conditions of the lemma hold for all MDPs of horizon up to  $H$  and extend it to MDPs of horizon  $H + 1$ .

Let M be of horizon  $H + 1$  and let  $A(\sigma) = \{a_1, \dots, a_k\}$  be the set of actions available at the starting state  $\sigma$ . Let  $R_i$  denote the set of all states that can be reached directly by taking action  $a_i$  on  $\sigma$ ; that is,  $R_j := \{s \in S : \mathcal{D}(\sigma, a_j, s) > 0\}.$  For each  $s \in R_j$ , we use  $\mathcal{M}_s$  to denote the subprocesses of  $\mathcal M$  that starts in state s and  $W^*_{\mathcal{M}_s}$  to denote the water filling surrogate cost of  $\mathcal{M}_s$ . By definition, each MDP  $\mathcal{M}_s$  has horizon up to H and thus by the induction hypothesis it admits a mapping  $\rho_s(\cdot)$  over its terminal states  $T(s)$  and a non-negative cost sharing vector  $b^s$  satisfying the properties of the lemma.

We will define the surrogate cost function  $\rho$  in two steps. First we will compute the water filling amortization of each action  $a_j$ . Let  $Z_j$  denote the random variable that first draws a state s from the distribution  $\mathcal{D}(\sigma, a_j, \cdot)$  and then draws a value from the distribution  $W^*_{\mathcal{M}_s}$ . Let  $g_j$  be the solution to the equation

<span id="page-12-0"></span>
$$
c(a_j) + \mathbb{E}[Z_j] = \mathbb{E}[\max\{g_j, Z_j\}].
$$

Then,  $\hat{Z}_j := \max\{g_j, Z_j\}$  is the water filling surrogate cost of the action  $a_j$ . We observe that for any  $y \ge g_j$ , we have  $c(a_j) + \text{E}[\min\{y, Z_j\}] = \text{E}[\min\{y, \hat{Z}_j\}]$ , and because  $\hat{Z}_j$  is always at least  $g_j$ , we get for all y:

$$
\min\{y, c(a_j) + \mathbb{E}[\min\{y, Z_j\}]\} = \mathbb{E}\left[\min\{y, \hat{Z}_j\}\right].
$$
\n(2)

We can now write the optimality curve of  $\mathcal M$  as:

$$
f_{\mathcal{M}}(y) = \min \left\{ y, \min_{j \in [k]} \left( c(a_j) + \sum_{s \in R_j} \mathcal{D}(\sigma, a_j, s) \cdot f_{\mathcal{M}_s}(y) \right) \right\}
$$

$$
= \min \left\{ y, \min_{j \in [k]} \left( c(a_j) + \sum_{s \in R_j} \mathcal{D}(\sigma, a_j, s) \cdot \mathcal{E} \left[ \min \{ y, W^*_{\mathcal{M}_s} \} \right] \right) \right\}
$$
  

$$
= \min \left\{ y, \min_{j \in [k]} \left( c(a_j) + \mathcal{E} \left[ \min \{ y, Z_j \} \right] \right) \right\}
$$
  

$$
= \min \left\{ y, \min_{j \in [k]} \mathcal{E} \left[ \min \left( y, \hat{Z}_j \right) \right] \right\}.
$$

Here the first equation follows from noting that in the local game  $(M, y)$ , the algorithm can either choose the outside option y or takes one of the actions  $a_j$  from  $\sigma$  and then proceeds optimally in the game  $(\mathcal{M}_s, y)$ where s is instantiated from  $R_j$ . The second equation follows by the definition of  $W^*_{\mathcal{M}_s}$ ; the third by the definition of  $Z_j$ ; and the fourth by Equation [\(2\)](#page-12-0).

Recall that  $f_{\mathcal{M}}(y) = \mathbb{E} \left[ \min\{y, W_{\mathcal{M}}^*\}\right]$ , and so, we conclude that

$$
\mathrm{E}\left[\min\{y, W_{\mathcal{M}}^*\}\right] \leq \mathrm{E}\left[\min\{y, \hat{Z}_j\}\right]
$$

for all  $j \in [k]$ . This implies that the random variable  $\hat{Z}_j$  second-order stochastically dominates the random variable  $W^*_{\mathcal{M}}$ , allowing us to use the following lemma.

<span id="page-13-0"></span>**Lemma 3.5.** (Second Order Stochastic Dominance.) Let  $X, Z$  be discrete random variables that satisfy the property  $E[\min\{y, X\}] \le E[\min\{y, Z\}]$  for all  $y \in \mathbb{R}$ . There exists a mapping  $m : \text{supp}(Z) \mapsto \Delta(\text{supp}(X))$ from the support of  $Z$  to distributions over the support of  $X$  such that:

- 1. X is obtained by sampling from  $m(z)$  for a randomly sampled  $z \sim Z$ .
- 2. For all  $z \in \text{support}(Z)$ , it holds that  $E[m(z)] \leq z$ .

We note that the lemma is standard (see for example [\[Strassen, 1965,](#page-35-15) [Föllmer and Schied, 2016\]](#page-34-19)) but we provide a constructive proof in Appendix [A](#page-36-0) for the sake of intuition and completeness. We apply Lemma [3.5](#page-13-0) to all tuples  $(W^*_{\mathcal{M}}, \hat{Z}_j)$  to obtain mappings  $m_j(\cdot)$ .

We are finally ready to define the amortized cost function  $\rho$  and the cost sharing vector b. Fix some  $t \in T$ , and let  $j \in [k]$  and  $s \in S$  be the indices of the unique action  $a_j$  and state  $s \in R_j$  such that  $t \in T(s)$ . We define

$$
\rho(t) := m_j(\max\{g_j, \rho_s(t)\})
$$

and

$$
b_{\sigma t} := \mathcal{E}\left[\rho(t)\right] - \mathcal{E}\left[\rho_s(t)\right]
$$

and for all other  $s' \in S \setminus \{\sigma\}$  with  $t \in T(s')$ , we use the same cost share  $b_{s't} = b_{s't}^s$  that was used in  $\mathcal{M}_s$ .

We will now show that these costs satisfy the properties claimed in the lemma. Fix any committing policy  $\pi \in \mathcal{C}(\mathcal{M})$ . Without loss, we will assume that  $\pi$  deterministically commits to a fixed action  $a_i$  at  $\sigma$ ; once our results are established for these committing policies, they immediately extend to arbitrary committing policies by linearity of expectation.

- Action independence: Drawing a terminal node  $t \sim p_{\pi}$  is equivalent to first drawing a state  $s \in R_i$ from the distribution  $\mathcal{D}(\sigma, a_j, \cdot)$  and then drawing a terminal node from the restriction of  $\pi$  to  $\mathcal{M}_s$ . Consider drawing t in this manner and then sampling from the distribution  $\rho_s(t)$ . By the induction hypothesis and the definition of  $Z_j$ , this provides us with a sample drawn from  $Z_j$ . Then,  $\max\{g_j, \rho_s(t)\}\$ with t drawn in this manner corresponds to an instantiation of  $\hat{Z}_j$ , and  $m_j$  applied to that instantiation results in an instantiation of  $W^*_{\mathcal{M}}$  by the definition of  $m_j$  and property (1) in Lemma [3.5.](#page-13-0)
- Cost sharing: This holds trivially by the definition of the cost shares in the starting state  $\sigma$ , the fact that we don't change the cost shares in any other state  $s \neq \sigma$  and the induction hypothesis.
- Cost dominance: By the induction hypothesis, the inequality holds for all nodes other than  $\sigma$ . Note that  $p_{\pi}$  only places non-zero probability mass on states downstream from action  $a_j$ . Denote by  $\pi_{|s}$  the

policy  $\pi$  confined to the subprocess  $\mu_s$  for some  $s \in R_j$ . Then we can write the total cost share paid to  $\sigma$  under some policy  $\pi \in \mathcal{C}(\mathcal{M})$  as:

$$
\sum_{t \in T_{\pi}} p_{\pi}(t) b_{\sigma t} = \sum_{s \in R_j, t \in T_s} p_{\pi}(t) (\mathbf{E} [\rho(t)] - \mathbf{E} [\rho_s(t)])
$$
\n
$$
= \sum_{t \in T_{\pi}} p_{\pi}(t) \mathbf{E} [\rho(t)] - \sum_{s \in R_j} \mathcal{D}(\sigma, a_j, s) \sum_{t \in T_{\pi|s}} p_{\pi|s}(t) \mathbf{E} [\rho_s(t)]
$$
\n
$$
= \mathbf{E} [W^*_{\mathcal{M}}] - \sum_{s \in R_j} \mathcal{D}(\sigma, a_j, s) \mathbf{E} [W^*_{\mathcal{M}_s}]
$$
\n
$$
\leq \mathbf{E} [\hat{Z}_j] - \mathbf{E} [Z_j] = c(a_j)
$$

which proves the claim by noting that  $p_{\pi}(\sigma) = 1$ . Here the first equation follows from the definition of  $b_{st}$ ; the second just rewrites the terms separately; the third is by the action independence of  $\rho$  as proved above, and by the induction hypothesis applied to  $\rho_s$  similarly; the fourth uses property 2. in Lemma [3.5](#page-13-0) for the first term and the definition of  $Z_j$  for the second term; and the last equality follows from the definition of  $\hat{Z}_j$ .

This concludes the proof of Lemma [3.3.](#page-11-1)

### <span id="page-15-0"></span>4 Local Approximation and Composition Theorems

In Section [3](#page-9-0) we showed that water filling surrogate costs provide a lower bound on the optimal cost for any instance  $\mathbb{I} = (\mathcal{M}_1, \cdots, \mathcal{M}_n, \mathcal{F})$  of the matroid-min-CICS. Furthermore, when all of the MDPs  $\mathcal{M}_i$  in the instance are Markov chains, not only is this lower bound exact but it also suggests a simple optimal policy. We will now put these concepts together to enable approximations for general instances. Following previous work on Pandora's Box, our approach finds a committing policy  $\pi_i \in \mathcal{C}(\mathcal{M}_i)$  for each MDP  $\mathcal{M}_i$ , that exhibits some notion of approximation in the local game for the MDP. Our goal is to then combine these local guarantees into a global approximation guarantee.

A natural first attempt is to find a committing policy  $\pi$  for M that achieves approximation through optimality curves:  $f_{\mathcal{M}^{\pi}}(y) \leq \alpha \cdot f_{\mathcal{M}}(y)$  for all y and some fixed factor  $\alpha \geq 1$ . As we showed in the introduction, however, this condition is not sufficient for a global guarantee. We adopt the following strengthening from [Scully and Doval](#page-35-1) [\[2024\]](#page-35-1).

<span id="page-15-1"></span>**Definition 7** (Local Approximation). Let M be any MDP. We say that a committing policy  $\pi \in \mathcal{C}(\mathcal{M})$  $\alpha$ -locally approximates M for some  $\alpha \geq 1$  if

$$
f_{\mathcal{M}^{\pi}}(\alpha y) \leq \alpha \cdot f_{\mathcal{M}}(y) \quad \forall y \in \mathbb{R}.
$$

We say that a decision process M admits an  $\alpha$ -local approximation if there exists a committing policy  $\pi \in \mathcal{C}(\mathcal{M})$  that  $\alpha$ -locally approximates it.

Local approximation has an intuitive interpretation. From the inherent connection between optimality curves and water filling surrogate costs of Definition [6,](#page-11-3) we can equivalently state the local approximation condition as

$$
\mathrm{E}\left[\min\{y,W_{\mathcal{M}^{\pi}}^*\}\right] \leq \mathrm{E}\left[\min\{y,\alpha\cdot W_{\mathcal{M}}^*\}\right] \;\;\forall y \in \mathbb{R}.
$$

By definition, water filling surrogate costs scale with the parameters of the underlying MDP; in other words, the surrogate cost of a scaled-up version of  $M$ , where all action costs and values are multiplied by α, will be precisely  $\alpha \cdot W_{\mathcal{M}}^*$ ; we use  $\alpha \mathcal{M}$  to denote this scaled-up version of M. With this in mind, the local approximation condition states that an optimal policy for the local game  $(M, y)$  under commitment  $\pi$  performs better than the optimal adaptive policy in the local game  $(\alpha \mathcal{M}, y)$ . Equivalently, the surrogate costs  $\alpha W^*_{\mathcal{M}}$  second-order stochastically dominate the surrogate costs  $W^*_{\mathcal{M}_\pi}.$ 

We can also establish local approximation through a stronger but conceptually easier relationship between the surrogate costs: namely, that  $\alpha W^*_{\mathcal{M}}$  first-order stochastically dominates  $W^*_{\mathcal{M}_\pi}$ . Formally, we define the following alternate notion of approximation, where for a quantile  $q \in [0,1]$  and a random variable X, we let  $X(q)$  denote the qth quantile value of the random variable:  $X(q) = \inf\{x : Pr[X \leq x] \geq q\}.$ 

**Definition 8** (Pointwise Approximation of Surrogate Costs). Let  $M$  be any MDP. We say that a committing policy  $\pi \in \mathcal{C}(\mathcal{M})$  α-pointwise approximates M for some  $\alpha \geq 1$  if we have  $W^*_{\mathcal{M}}(\pi(q)) \leq \alpha \cdot W^*_{\mathcal{M}}(q)$  for all  $q \in [0, 1].$ 

<span id="page-15-2"></span>The following is immediate, but the converse is not always true.

Fact 2. For any MDP M, any policy that  $\alpha$ -pointwise approximates M will also  $\alpha$ -locally approximate M.

#### Composition of Local Approximation

It is worth noting that in the special case of  $\alpha = 1$ , local approximation coincides with Whittle's condition [\[Whittle, 1980,](#page-35-3) [Glazebrook, 1982\]](#page-35-9), according to which an MDP M admits a fixed sequence of actions that is always optimal for the local game  $(M, y)$ . In that sense, local approximation can also be viewed as a relaxation of Whittle's condition. It is well-known that if all the MDPs satisfy Whittle's condition, then we can efficiently find an optimal solution to matroid-min-CICS. In other words, if each  $\mathcal{M}_i$  admits a 1-local approximation under some local commitment  $\pi_i \in \mathcal{C}(\mathcal{M}_i)$ , then we can deduce that the global commitment  $\Pi = (\pi_1, \dots, \pi_n)$  allows for a 1-approximation with respect to the optimal adaptive policy. We refer to such statements as *composition results*, since they allow us to combine local guarantees on the underlying MDPs into global approximation guarantees for matroid-min-CICS.

The main strength of local approximation lies in the fact that much like Whittle's condition, it satisfies the following composition result. The proof is straightforward from our discussion above; see Appendix [B.](#page-46-0)

<span id="page-16-1"></span>**Theorem 4.1.** Let  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  be any instance of matroid-min-CICS, where each MDP  $\mathcal{M}_i$ admits an  $\alpha$ -local approximation under some committing policy  $\pi_i \in \mathcal{C}(\mathcal{M}_i)$ . Then,

$$
\mathbf{E}\left[\min_{S\in\mathcal{F}}\sum_{i\in S}W_{\mathcal{M}_i^{\pi_i}}^*\right] \leq \alpha \cdot \mathbf{E}\left[\min_{S\in\mathcal{F}}\sum_{i\in S}W_{\mathcal{M}_i}^*\right].
$$

Observe that Theorem [F.2](#page-52-1) directly implies a way to efficiently approximate the optimal adaptive policy for any instance of matroid-min-CICS, assuming that the local approximation guarantees are met. We state this approach in Algorithm [1.](#page-16-0)

<span id="page-16-0"></span>

Clearly, all the steps of the algorithm run in polynomial time to the size of the MDPs. By combining the optimality of the water filling index policy on Markov chains (Theorem [3.2\)](#page-10-2) with our lower bound on the optimal adaptive policy (Theorem [3.4\)](#page-11-4) and our composition theorem (Theorem [4.1\)](#page-16-1), we immediately obtain the following:

<span id="page-16-2"></span>**Corollary 4.2.** Let  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  be any instance of matroid-min-CICS. Instantiating Algorithm [1](#page-16-0) with a set of committing policies  $\pi_i$  that  $\alpha$ -locally approximate the MDPs  $\mathcal{M}_i$  results to an  $\alpha$ -approximation to the optimal adaptive policy for I.

### <span id="page-17-0"></span>5 Pandora's Box with Partial Inspection (Minimization)

In classical Pandora's Box, the algorithm pays some cost  $c^o$  to open a box and observe its value, and is then allowed to select the box. We will now study the minimization version of the generalization called Pandora's Box with Partial Inspection (henceforth, PBPI) where the algorithm can additionally "peek" into the box at a smaller cost  $c^p < c^o$  and learn its value. If upon obtaining this information, the algorithm wants to select the box, it must still open the box at a cost of  $c^o$  before accepting it. This presents a choice: in some cases it may be better to open the box outright, while in others it is better to pay the smaller peeking cost to potentially avoid paying the opening cost later.

Formally, we consider an instance with n partial inspection boxes (henceforth, PI-boxes)  $\{\mathcal{B}_i\}_{i=1}^n$ , where each box  $\mathcal{B}_i = (\mathcal{D}_i, c_i^o, c_i^p)$  is characterized by a distribution  $\mathcal{D}_i$  over values; an opening cost  $c_i^o > 0$ ; and a peeking cost  $c_i^p \in (0, c_i^o)^6$  $c_i^p \in (0, c_i^o)^6$ . Each PI-box  $\mathcal{B}_i$  can be expressed as a Costly Information MDP with two possible actions, peeking and opening, where we can interpret the accept action after peeking as incurring a cost of  $c<sup>o</sup>$ . An instance of PBPI corresponds to an instance of min-CICS over the corresponding MDPs. A pictorial representation of the different states of a box and the underlying MDP is shown in Figure [3.](#page-17-3)

<span id="page-17-3"></span>

Figure 3: Each PI-box  $\mathcal{B} = (\mathcal{D}, c^o, c^p)$  is initially closed. In order to learn the value realization realization  $X \sim \mathcal{D}$ , the decision maker can either peek into the box (at a cost of  $c^p$ ) or open it (at a cost of  $c^o$ ). To accept the box, the decision maker must first open it and then pay its (now known) value  $X$ .

Our Results. In Section [5.1](#page-17-1) we will show that while there are simple instances of PBPI where all committing algorithms are sub-optimal up to a constant factor (Example [2\)](#page-17-4), there exists a trivial committing algorithm that achieves a  $\phi \approx 1.618$  approximation to the optimal adaptive policy (Lemma [5.2\)](#page-18-1). In order to aigorithm that achieves a  $\phi \approx 1.618$  approximation to the optimal adaptive policy (Lemma [5.2](#page-18-0)). In order to further improve on this guarantee, in Section 5.2 we will develop a  $\sqrt{2} \approx 1.414$  local approximation guarante for any PI-box  $\beta$  (Lemma [5.3\)](#page-18-2); combined with the meta-algorithm of Corollary [4.2,](#page-16-2) this directly implies our main result for this section:

<span id="page-17-5"></span>**Theorem 5.1.** There exists an efficient algorithm that achieves a  $\sqrt{2}$ -approximation to the optimal adaptive policy for any instance of matroid-PBPI.

#### <span id="page-17-1"></span>5.1 Lower and Upper Bounds on the Adaptivity Gap of PBPI

In this section, we take a closer look at committing algorithms for PBPI and the approximation guarantees that they can achieve with respect to the optimal adaptive policy. Observe that a committing algorithm for PBPI needs to decide in advance (perhaps randomly) whether each box  $\mathcal{B}_i$  will be directly opened or peeked into and then (potentially) opened. We begin by showing that that there are simple instances where all committing policies are sub-optimal up to a constant factor.

<span id="page-17-4"></span>Example 2. Consider an instance of single-selection PBPI over two boxes. The first box has an opening cost of 1, a peeking cost of  $\frac{1}{4}$  and its random value is 0 with probability  $\frac{1}{2}$  and 2 otherwise. The second box has both an opening and peeking cost of 0, and its random value is 2 with probability  $\frac{1}{2}$  and  $\infty$  otherwise.

<span id="page-17-2"></span><sup>&</sup>lt;sup>6</sup>Since we can only select an opened box, if  $c_i^p \geq c_i^o$ , then we can safely exclude the peeking action and the box reduces to a classical Pandora's Box.

- Since opening the second box is free, we can assume without loss that all policies start by opening it and observing its value, call it y. The optimal policy will simply open and accept the first box if  $y = \infty$ . But if  $y = 2$ , it can peek into the first box and only open it if it contains a value of 0. The expected cost of this algorithm is  $\frac{15}{8}$ .
- Now consider any policy that commits to opening or peeking into box 1 before observing y. If it commits to opening the first box, it accepts the value of this box regardless of  $y$ , as box 1 always has a value smaller than y. The expected cost of this policy is 2. If it commits to peeking into the first box, then at  $y = \infty$  it incurs a cost of  $\frac{9}{4}$  due to having to pay the extra peeking cost; and at  $y = 2$  it incurs an expected cost of  $\frac{7}{4}$  by opening the first box only if it contains a value of 0. It's net cost is again 2.

Since both deterministic committing policies have an expected cost of 2, so does any randomized commitment. Consequently, the expected cost of the optimal adaptive policy is strictly smaller than that of the optimal committing policy.

Example [2](#page-17-4) illustrates that in general, committing algorithms are worse than the optimal adaptive policy by a factor of at least  $\frac{16}{15}$ . On the positive side, it is easy to obtain a 2-approximation via committing algorithms. In particular, consider the policy that always commits to peeking. This policy can mimic the optimal one as follows. Whenever the optimal algorithm peeks, so does this committing policy. Whenever the optimal algorithm opens without peeking, the committing policy peeks and then opens; on a box with costs  $c_i^p$  and  $c_i^o$ , this policy pays  $c_i^p + c_i^o$  or at most twice the amount  $c_i^o$  paid by the optimal algorithm. In fact, we can further refine this argument by choosing the action we commit to more carefully: there always exists a simple commitment under which we can achieve a  $\phi \approx 1.618$  approximation to the optimal adaptive policy. The proof is presented in Appendix [C.](#page-47-0)

<span id="page-18-1"></span>**Lemma 5.2.** Consider any instance  $\mathbb{I} = (\mathcal{B}_1, \dots, \mathcal{B}_n, \mathcal{F})$  of matroid-PBPI and partition the n boxes into two sets

$$
O:=\left\{i\in[n]:\frac{c_i^o}{c_i^p}\leq 1+\frac{c_i^p}{c_i^o}\right\}
$$

and  $P = [n] \setminus O$ . The policy that commits to directly opening the boxes in O and peeking before opening the boxes in P achieves a  $\phi$ -approximation to the optimal adaptive policy.

We note that by using a *global argument* such as the one above, i.e. an argument where we charge each action of the committing algorithm directly to the optimal adaptive policy, one cannot improve on this upper bound of  $\phi$  for the gap between committing and adaptive policies (for example, consider a setting where all the boxes have  $c^p = 1$  and  $c^o = \phi$ ). In order to improve on this guarantee, we would need to leverage our knowledge of the value distributions. In the next section, we achieve this by establishing local approximation guarantees for PBPI.

#### <span id="page-18-0"></span>5.2 Local Approximation Guarantees for PBPI

In this section, we prove Theorem [5.1](#page-17-5) by providing a  $\sqrt{2}$ -local approximation for any PI-box, as stated below. Given a PI-box  $\mathcal{B} = (\mathcal{D}, c^o, c^p)$ , let  $g^p$  denote the water filling (a.k.a. Gittins) index of the policy that commits to peeking. Equivalently,  $g^p$  is the solution to the equation  $c^p = \mathbb{E}_{X \sim \mathcal{D}} [(g^p - X - c^o)^+]$ . We call  $g^p$ the peeking index of the box.

<span id="page-18-2"></span>**Lemma 5.3.** Let  $\mathcal{B} = (\mathcal{D}, c^o, c^p)$  be a PI box with peeking index  $g^p$ . Let  $\pi$  be the policy that commits to the opening action whenever

$$
\frac{c^o}{c^p} \cdot \left(1 - \frac{c^o}{g^p}\right) \le 1 + \min\left(\frac{c^p}{c^o}, \frac{c^o}{g^p}\right)
$$

and to the peeking action otherwise. Then,  $\pi$  is a  $\sqrt{2}$ -local approximation to B.

The rest of this section is devoted to proving Lemma [5.3.](#page-18-2) We first note the following characterization:

**Definition 9.** The optimality curve of a PI-box  $\mathcal{B} = (\mathcal{D}, c^o, c^p)$  for cost minimization is given by

$$
f_{\mathcal{B}}(y) := \min\{y, f_{\mathcal{B}}^o(y), f_{\mathcal{B}}^p(y)\}\
$$

where  $f_{\mathcal{B}}^o(y) := c^o + \mathbb{E}_{X \sim \mathcal{D}} \left[ \min\{y, X\} \right]$  is the optimality curve of the policy that commits to opening the box and  $f_{\mathcal{B}}^p(y) := c^p + \mathbb{E}_{X \sim \mathcal{D}} \left[ \min\{y, X + c^o\} \right]$  is the optimality curve of the policy that commits to peeking. We also define the **opening index** of the box,  $g^o$ , as the water filling index of the opening policy, equivalently, the solution to the equation  $c^o = \mathbb{E}_{X \sim \mathcal{D}}[(g^o - X)^+]$ . We depict these curves and indices in Figure [4.](#page-19-0)

<span id="page-19-0"></span>

Figure 4: The optimality curves for a PI-box  $\beta$  with opening cost  $c^o = 0.5$ , value  $X = 0$  with probability 0.5 and  $X = 2$  with probability 0.5 and peeking cost  $c^p = 0.4$  (left) and  $c^p = 0.1$  (right). The optimality curve of  $\beta$  is given by the minimum of the three curves. Observe that the curves of the opening and peeking action intersect at a unique point  $\tau$  and that unless  $g^p < g^o$ , the opening action dominates the peeking action.

Observe that if the optimal adaptive policy prefers opening over peeking the box for some outside option y, then the same will be true for any  $y' > y$ . Also, for  $y = 0$  peeking is clearly preferable to opening as  $c^p < c^o$ . Thus, the optimality curves  $f^o_{\mathcal{B}}(y)$  and  $f^p_{\mathcal{B}}(y)$  will have a unique intersection (up to an interval). We use  $\tau$  to denote this intersection; formally,  $\tau$  is the maximal solution to equation

$$
E_{X \sim D} [(\tau - X)^+] - E_{X \sim D} [(\tau - X - c^o)^+] = c^o - c^p.
$$

This implies that unless  $g^p < g^o < \tau$ , the opening action will dominate the peeking action for all values of the outside option  $y$  for which accepting it isn't optimal; as a consequence, such instances trivially admit a 1-local approximation by committing to the opening action. From now on, we assume that  $g^p < g^o < \tau$ . The, Lemma [5.3](#page-18-2) is a consequence of the following claims, establishing local approximation guarantees for the opening and peeking actions respectively.

<span id="page-19-1"></span>**Claim 1.** For all  $y \in \mathbb{R}$ , it holds that  $\min\{y, f^o_{\mathcal{B}}(y)\} \leq \alpha \cdot f_{\mathcal{B}}(\frac{y}{\alpha})$  for  $\alpha = \frac{c^o}{c^p}$  $\frac{c^o}{c^p} \cdot \left(1 - \frac{c^o}{g^p}\right)$  $\frac{c^o}{g^p}\Big).$ 

<span id="page-19-2"></span>**Claim 2.** For all  $y \in \mathbb{R}$ , it holds that  $\min\{y, f^p_{\mathcal{B}}(y)\} \leq \alpha \cdot f_{\mathcal{B}}(\frac{y}{\alpha})$  for  $\alpha = 1 + \min\left(\frac{c^p}{c^{\alpha}}\right)$  $\frac{c^p}{c^o}, \frac{c^o}{g^p}$  $\frac{c^o}{g^p}\bigg).$ 

Note that once proven, they immediately imply an  $\alpha$ -local approximation with

$$
\alpha = \min \left\{ \frac{c^o}{c^p} \cdot \left( 1 - \frac{c^o}{g^p} \right) , \ 1 + \frac{c^p}{c^o} , \ 1 + \frac{c^o}{g^p} \right\}.
$$

Fixing c<sup>o</sup> and c<sup>p</sup> and letting  $\lambda := c^p/c^o \in (0,1)$ , the minimum of the first and third terms is maximized at Fixing c and c and letting  $\lambda := c^r/c^r \in (0, 1)$ , the  $g^p = c^o \cdot \frac{1+\lambda}{1-\lambda}$ ; so we have  $\alpha \le \min\{1+\lambda, \frac{2}{1+\lambda}\} \le \sqrt{2}$ .

**Proof of Claim [1.](#page-19-1)** By concavity, we have that  $f_{\mathcal{B}}^o(y) \leq \alpha f_{\mathcal{B}}^o(\frac{y}{\alpha})$  for all  $y \in \mathbb{R}$  and  $\alpha \geq 1$ ; thus, we only need to verify the condition for y such that  $f_{\mathcal{B}}(\frac{y}{\alpha}) = f_{\mathcal{B}}^p(\frac{y}{\alpha})$ . In other words, we only need to verify that  $\min\{y, f_{\mathcal{B}}^o(y)\}\leq \alpha f_{\mathcal{B}}^p(\frac{y}{\alpha})$  for all  $y\in [\alpha g^p, \alpha\tau]$ .

For start, we will show that the condition holds for  $\alpha = g^o/g^p$ ; notice that for this parameter,  $f^o_B(y) \leq y$ for all  $y \in [\alpha g^p, \alpha \tau]$  and thus we can now write our condition as  $D_\alpha(y) := \alpha f^p_{\mathcal{B}}(\frac{y}{\alpha}) - f^o_{\mathcal{B}}(y) \geq 0$  for all  $y \in [\alpha g^p, \alpha \tau]$ . Let  $F(\cdot)$  denote the CDF of distribution D. By definition of  $f^o_B(y)$  and  $f^p_B(y)$ , we have

$$
\frac{d}{dy}f^o_{\mathcal{B}}(y) = 1 - F(y) , \frac{d}{dy}f^p_{\mathcal{B}}(y) = 1 - F(y - c^o)
$$

and by taking the derivative of  $D_{\alpha}(\cdot)$ , we immediately obtain that  $D_{\alpha}(\cdot)$  is a weakly increasing function of y and thus the condition only needs to hold on  $y = \alpha g^p$ . Observe that for  $\alpha = g^o/g^p$  and  $y = \alpha g^p = g^o$ , we have  $D_{\alpha}(y) = \frac{g^{\circ}}{g^p}$  $\frac{g^o}{g^p} \cdot f^p_{\mathcal{B}}(g^p) - f^o_{\mathcal{B}}(g^o) = g^o - g^o = 0$  and thus we obtain that the opening action always achieves  $a \frac{g^o}{a^p}$  $\frac{g^2}{g^p}$ -local approximation. The proof is concluded by showing that

$$
\frac{g^o}{g^p} \le \frac{c^o}{c^p} \cdot \left(1 - \frac{c^o}{g^p}\right).
$$

For this purpose, we define the function  $h(z) := \mathbb{E}_{X \sim \mathcal{D}}[(z - X)^+]$  and observe that  $h'(z) = F(z)$ ; thus,  $h(z)$  is an increasing and convex function of z with  $h(0) = 0$ . Furthermore, by definition of the indices and the curves we have that  $h(g^o) = c^o$  and  $h(g^p - c^o) = c^p$ . Thus, by convexity, we immediately obtain that

$$
\frac{h(g^o)}{g^o} \geq \frac{h(g^p-c^o)}{g^p-c^o} \Rightarrow \frac{c^o}{g^o} \geq \frac{c^p}{g^p-c^o}
$$

from which the claim follows.

**Proof of Claim [2.](#page-19-2)** By concavity, we have that  $f_{\mathcal{B}}^p(y) \leq \alpha f_{\mathcal{B}}^p(\frac{y}{\alpha})$  for all  $y \in \mathbb{R}$  and  $\alpha \geq 1$ ; thus, we only need to verify the condition for y such that  $f_{\mathcal{B}}(\frac{y}{\alpha}) = f_{\mathcal{B}}^o(\frac{y}{\alpha})$ ; notice that this corresponds to  $y \ge \alpha \tau > g^p$  and thus  $f_{\mathcal{B}}^p(y) \leq y$ . In other words, we only need to verify  $f_{\mathcal{B}}^p(y) \leq \alpha f_{\mathcal{B}}^o(\frac{y}{\alpha})$  forall  $y \geq \alpha \tau$ .

Let  $D_{\alpha}(y) := \alpha \cdot f_{\mathcal{B}}^o(y/\alpha) - f_{\mathcal{B}}^p(y)$ . We need  $D_{\alpha}(y) \ge 0$  for  $y \ge \alpha \tau$ . The condition immediately holds at  $y = \alpha \tau$  since

$$
\alpha f_{\mathcal{B}}^{o}(\frac{\alpha \tau}{\alpha}) = \alpha f_{\mathcal{B}}^{o}(\tau) = \alpha f_{\mathcal{B}}^{p}(\tau) \ge f_{\mathcal{B}}^{p}(\alpha \tau)
$$

by concavity of  $f_{\mathcal{B}}^p(\cdot)$ . Next, observe that

$$
\frac{d}{dy}D_{\alpha}(y) = \frac{d}{dy}f_{\beta}^o(\frac{y}{\alpha}) - \frac{d}{dy}f_{\beta}^p(y) = F(y - c^o) - F(\frac{y}{\alpha})
$$

and thus  $D_a(y)$  gets minimized at  $y = \frac{\alpha \cdot c^{\circ}}{\alpha - 1}$  $\frac{\alpha \cdot c^o}{\alpha-1}$ . If  $\frac{\alpha \cdot c^o}{\alpha-1} \leq \alpha \tau$  or equivalently  $\alpha \geq 1 + c^o/\tau$ , then  $\frac{d}{dy} D_{\alpha}(y) \geq 0$ in the area of interest; thus, we obtain that the peeking action always achieves a  $1 + \frac{c^o}{\tau} \leq 1 + \frac{c^o}{g^p}$  $rac{c^{\circ}}{g^p}$  local approximation. To complete the proof, we need to show that the peeking action also achieves a  $1 + \frac{c^p}{c^q}$  $\frac{c^{\nu}}{c^{\sigma}}$  local approximation or equivalently that for  $\alpha = 1 + \frac{c^p}{c^q}$  $\frac{c^p}{c^o}$  we have that  $\min_{y \ge \alpha \tau} D_\alpha(y) = D(\frac{\alpha \cdot c^o}{\alpha - 1})$  $\frac{\alpha \cdot c^{\circ}}{\alpha - 1}$ )  $\geq 0$ .

Once again, we consider the function  $h(z) = \mathbb{E}_{X \sim \mathcal{D}}[(z - X)^+]$ ; this time, we observe that by definition we have

$$
f_{\mathcal{B}}^o(y) = c^o + y - h(y)
$$
,  $f_{\mathcal{B}}^p(y) = c^p + y - h(y - c^o)$ .

From this, the condition  $D(\frac{\alpha \cdot c^{\circ}}{\alpha-1})$  $\frac{\alpha \cdot c^{\circ}}{\alpha - 1}$ ) ≥ 0 translates to

$$
h(\frac{c^o}{\alpha - 1}) \le \frac{\alpha \cdot c^o - c^p}{\alpha - 1}
$$

and for  $\alpha = 1 + \frac{c^p}{c^q}$  $\frac{c^{\rho}}{c^{\rho}}$ , this statement is equivalent to

$$
h(\frac{c^o \cdot c^o}{c^p}) \le \frac{c^o \cdot c^o}{c^p}
$$

which is clearly true, by definition of  $h(\cdot)$ .

## <span id="page-21-0"></span>6 The Weighing Scale Problem (Minimization)

We will now introduce the Weighing Scale (henceforth, WS) problem. A decision maker is presented with  $n$ alternatives  $(X_i, c_i)$  and a combinatorial constraint  $\mathcal{F} \subseteq 2^{[n]}$ ;  $X_i \geq 0$  is the random value of the alternative realized independently by a known distribution and  $c_i \geq 0$  is a weighing cost. The only way the decision maker can determine any further information about each value  $X_i$  beyond its distribution, is to use a weighing scale to compare it against some fixed threshold t of their choosing at the additional cost of  $c_i$  and learn whether  $X_i \leq t$  or not. The process terminates with the decision maker selecting a feasible set of alternatives  $S \subseteq \mathcal{F}$ , paying their total value.

Note that each alternative  $(X_i, c_i)$  corresponds to a Costly Information MDP  $\mathcal{M}_i$  that captures the information acquisition process described above. In particular, the available actions at the starting state of each MDP  $\mathcal{M}_i$  are as follows:

- 1. Pick a threshold  $t \in \text{support}(X_i)$  and weigh the alternative against it at a cost of  $c_i$ . Upon taking one of these actions, the MDP advances to one of two random subprocesses  $\mathcal{M}_i^{\leq t}$  and  $\mathcal{M}_i^{>t}$ , defined over the random variables  $(X_i|X_i \leq t)$  and  $(X_i|X_i > t)$  respectively, based on the random outcome of the weighing.
- 2. Commit no more weighings of the alternative; this is a 0-cost action resulting to a terminal state  $x_i$  of value  $v(x_i) := E[X_i].$

From this equivalence, an instance of WS corresponds to an instance of min-CICS over the corresponding MDPs.[7](#page-21-1) Our main result for this section is the following.

<span id="page-21-2"></span>**Theorem 6.1.** For any instance  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  of matroid-WS, there exists an efficient algorithm that achieves an  $O(\max_i \kappa_i)$ -approximation to the optimal adaptive policy, with the parameter  $\kappa_i$  for each alternative  $i \in [n]$  defined as

$$
\kappa_i := \frac{\mu_i}{M_i} + \log \frac{\mu_i}{g_i}
$$

where  $\mu_i = \mathbb{E}[X_i]$  is the expected value of  $X_i$ ,  $M_i$  is the median value of  $X_i$ , and  $g_i$  denotes the Gittins index of the alternative; i.e. the solution to the equation  $c_i = \mathbb{E}[(g_i - X_i)^+]$ .

We prove Theorem [6.1](#page-21-2) by showing that for each alternative  $i \in [n]$ , the MDP  $\mathcal{M}_i$  admits an  $O(\kappa_i)$ pointwise approximation. We note that without any assumptions on the distributions of the values, the parameters  $\kappa_i$  can be unbounded; in particular, for heavy tail distributions,  $\mu_i/M_i$  can be arbitrarily large. We show in Theorem [6.2](#page-21-3) (proven in Appendix [D\)](#page-48-0) that this dependence is necessary for any pointwise approximation guarantee. Better bounds may be possible via the weaker notion of local approximation or a global argument.

<span id="page-21-3"></span>**Theorem 6.2.** For any  $\alpha \geq 1$ , there exists a WS alternative M that does not admit an  $\alpha$ -pointwise approximation.

**Proof of Theorem [6.1.](#page-21-2)** We begin by introducing a committing policy for WS described below, which we call the One-Sided Halving algorithm. The policy begins by weighing the alternative against some threshold  $t_2$ ; if the alternative is larger, then the policy commits to no more weighings and if it is smaller, it halves its threshold from  $t_2$  to  $t_2/2$  and repeats, until the threshold reaches some fixed lower bound  $t_1$ .

<span id="page-21-1"></span><sup>7</sup>Technically, our framework does not capture infinite horizon MDPs. However, observe that whenever the decision maker has identified that the value of the alternative lies in some interval of length  $\leq c$ , performing any extra weighings is suboptimal. Thus, the corresponding MDPs for the WS problem are finite horizon without loss.

Algorithm 2: One-Sided Halving Algorithm.

**Input:** An MDP M corresponding to an alternative  $(X, c)$  and two thresholds  $0 < t_1 < t_2$ . 1 Set  $t = t_2$ . 2 while  $t \geq t_1$  do  $\bullet$  Weigh the random variable X against t. 4 if  $X \leq t$  then  $t = t/2$  else break. 5 end <sup>6</sup> Commit to no more weighings for the alternative.

Let h be the solution to equation  $c = \mathbb{E}[(X-h)^+]$ . The following lemma, combined with our reduction from pointwise to local approximation (Fact [2\)](#page-15-2) and our local approximation composition algorithm (Corollary [4.2\)](#page-16-2), directly implies Theorem [6.1.](#page-21-2)

<span id="page-22-0"></span>**Lemma 6.3.** Committing to no weighings if  $g > min\{\mu, M\}$ , and otherwise committing to the One-Sided Halving algorithm with  $t_1 = g$  and  $t_2 = \min(M, h)$  achieves an  $O(\kappa)$ -pointwise approximation.

The rest of this section is dedicated towards proving Lemma [6.3.](#page-22-0) We fix a WS alternative  $\mathcal M$  corresponding to random value X of support  $\mathcal{X} := \text{support}(X)$  and weighing cost c. We use  $\mu$ , M, g, h and  $\kappa$  to denote the parameters of the alternative, as previously defined. In order to develop our pointwise approximation guarantees, we first need to obtain expressions for the surrogate costs.

**Surrogate cost of M.** We begin by noting that the optimality curve of  $M$  has a very simple form. Indeed, consider the local game  $(M, y)$ ; the optimal adaptive policy has only three choices available: either accept the outside option at a cost of  $y$ , or accept the alternative without performing any weighings at an expected cost of  $\mu$  or perform a single weighing of the alternative against y to determine which of the two costs is smaller. Thus, we immediately obtain that  $f_{\mathcal{M}}(y) = \min(y, \mu, c + \mathbb{E}[\min\{y, X\}]).$ 

Observe that by definition, g corresponds to the maximal threshold for which  $y \leq c+E$  [min{y, X}] for all  $y \leq g$ . Thus, if  $\mu \leq g$ , then the cost  $c + \mathbb{E}[\min\{y, X\}]$  will always be dominated by either y or  $\mu$ , making the weighing action universally sub-optimal. If that's the case, then we can safely commit to blindly accepting the alternative without performing any weighings, achieving a 1-local approximation. Thus, from now on we will be assuming that  $g < \mu$ . Recall that g satisfies  $c = \mathbb{E}[(g - X)^+]$ ; h satisfies  $c = \mathbb{E}[(X - h)^+]$ ; and  $\mu$ satisfies  $E[(\mu - X)^+] = E[(X - \mu)^+]$  by its definition. Then we can deduce that  $g < \mu$  implies  $\mu < h$ . We can therefore re-write the optimality curve of  $\mathcal M$  as

$$
f_{\mathcal{M}}(y) = \begin{cases} y & \text{if } y < g \\ \mathbb{E} \left[ \min(y, \max(g, X)) \right] & \text{if } y \in [g, h] \\ \mu & \text{if } y > h \end{cases}
$$

From this, we obtain the following characterization of the surrogate costs. We note that the same result is proven by [Scully and Doval](#page-35-1) [\[2024\]](#page-35-1), as the optimality curve for the alternative  $\mathcal M$  coincides with the optimality curve of a Pandora's Box with optional inspection in the minimization setting.

**Fact 3.** The water filling surrogate cost W<sup>\*</sup><sub>M</sub> of M corresponds to sampling  $x \sim X$  and returning

$$
\rho^*(x) := \min(h, \max(g, x)).
$$

Committing policies. A committing policy  $\pi \in \mathcal{C}(\mathcal{M})$  for WS will declare in advance a decision tree over pre-specified thresholds against which it will weigh X, resulting in a Markov chain  $\mathcal{M}^{\pi}$ . Importantly,  $\pi$ will end up partitioning the support  $X$  of distribution  $D$  into a set of intervals, corresponding to its terminal states. Furthermore, the probability of running the Markov chain  $\mathcal{M}^{\pi}$  and ending up in a terminal state t corresponding to some interval I will be precisely  $Pr[X \in I]$ . A pictorial representation of such Markov

chains is given in Figure [5.](#page-23-0) We use  $\mathcal{I}^{\pi}$  to denote this set of intervals, and  $t(I)$  to denote the terminal state of  $\mathcal{M}^{\pi}$  corresponding to interval  $I \in \mathcal{I}^{\pi}$ . Furthermore, the probability of running the Markov chain  $\mathcal{M}^{\pi}$ and ending up in a terminal state  $t(I)$  will be precisely  $Pr[X \in I]$ . This allows us to obtain the following characterization of surrogate costs for committing policies.

**Fact 4.** For any committing policy  $\pi \in \mathcal{C}(\mathcal{M})$ , the water filling surrogate cost  $W^*_{\mathcal{M}^{\pi}}$  corresponds to sampling  $x \sim X$  and returning

$$
\rho^{\pi}(x) := W^*_{\mathcal{M}^{\pi}}(t(I))
$$

<span id="page-23-0"></span>for the unique interval  $I \in \mathcal{I}^{\pi}$  that contains x.



Figure 5: The indices q and h of the alternative can be obtained by the inverse CDF of the random value; the highlighted ares equal the weighing cost c. The figure on the right corresponds to the Markov chain for One-Sided Halving instantiated with threshold  $t_1 = 2$ ,  $t_2 = 8$  for an instance with value  $X \sim \text{Unif}[0, 10]$ . The terminal states partition the support of  $X$ ; the probability and value at each terminal equals the probability and the conditional expectation of the corresponding interval.

We are now ready to prove Lemma [6.3.](#page-22-0) Let  $\pi$  be the committing policy described in Lemma 6.3. We have already handled the case of  $g > \mu$ . Next, consider  $g > M$ . In this case  $\pi$  commits to no weighings,  $\mathcal{M}^{\pi}$  is simply a terminal state of value  $\mu$  and thus  $\rho^{\pi}(x) = \mu$  for all  $x \in \mathcal{X}$ . Since  $\rho^*(x) \geq g$  for all  $x \in \mathcal{X}$ , this trivially implies a  $\alpha = \mu/g \leq \mu/M$  pointwise approximation and the lemma follows.

If  $M \geq g$ , then  $\pi$  corresponds to the one-sided halving algorithm with  $t_1 = g$  and  $t_2 = \min(M, h)$ . Observe that by definition, the minimum threshold used by the policy will be some  $t_f \in [g, 2g]$ . Thus, the first interval in  $\mathcal{I}^{\pi}$  will be  $I_0 = (-\infty, t_f]$ . Note that

$$
\Pr\left[X \in I_0\right] = \Pr\left[X \le t_f\right] \ge \Pr\left[X \ge g\right] \ge \frac{c}{g}
$$

where the last inequality follows from the fact that  $c = \mathbb{E}[(g - X)^+] \leq g \cdot \Pr[X \leq g]$ . Furthermore, recall that by the definition of (any) amortization, the cost shares for the amortization of a state  $s$  satisfy

$$
p(s) \cdot c(s) = \sum_{t \in T(S)} p(t) \cdot b_{st}
$$

and since  $Pr[X \in I_0] \ge c/g$  and all the action costs are  $c(s) = c$ , this implies that in any step during the amortization of  $\mathcal{M}^{\pi}$ , the surrogate cost of terminal  $t(I_0)$  increases by at most g. Finally, we note that the horizon of  $\mathcal{M}^{\pi}$  will be at most

$$
k := \log \frac{t_2}{t_f} \le \log \frac{M}{g} \le \log \frac{2\mu}{g}
$$

and thus we conclude that the total number of amortization steps will be k, and that the total increase in the value of the first terminal will be at most  $kq$ .

Up next, we will argue that the terminal  $t(I_0)$  will be the terminal that suffers the maximum increase during the water-filling amortization of  $\mathcal{M}^{\pi}$ . Note that by definition of  $\mathcal{M}^{\pi}$ , the initial value of a terminal

state  $t(I)$  corresponding to some  $I \in \mathcal{I}^{\pi}$  will be precisely  $\mu(I) := E[X|X \in I]$ . Thus, terminal  $t(I_0)$  starts with the minimum value. Furthermore, by the one-sided structure of  $\mathcal{M}^{\pi}$ , observe that  $t(I_0)$  is a terminal state for all intermediate action states, and thus it will participate in all the stages of the water filling amortization. These two facts immediately prove the claim.

From the above, we can summarize that for all  $I \in \mathcal{I}^{\pi}$  we have

$$
W^*_{\mathcal{M}^\pi}(t(I)) \le \mu(I) + k \cdot g
$$

and thus for any  $x \in \mathcal{X}$ , we have  $\rho^{\pi}(x) \leq \mu(I) + kg$  for the unique  $I \in \mathcal{I}^{\pi}$  for which  $x \in I$ . It remains to upper bound the expectations  $\mu(I)$ . For  $I_0 = (-\infty, t_f] \subseteq (-\infty, 2g]$  we have that  $\mu(I_0) \leq 2g$ . For the maximum interval  $I = (t_2, \infty)$ , we have that  $\mu(I) \leq 2\mu$ ; this is a consequence of the fact that  $t_2 \leq M$  and thus  $E[X|X > t_2] \leq E[X|X \geq M] \leq 2\mu$ . Finally, for any other interval I, we know that the ratio between its endpoints will be precisely 2 due to the halving and thus  $\mu(I) \leq 2x$  for any  $x \in I$ .

To summarize, we have shown that for all  $x \in \mathcal{X}$ :

$$
\rho^{\pi}(x) \le \begin{cases} kg + 2g & \text{if } x < g \\ kg + 2x & \text{if } x \in [g, t_2] \\ kg + 2\mu & \text{if } x > t_2 \end{cases}
$$

Observe that since  $t_2 \leq M \leq 2\mu$ , we have that for any  $x \in \mathcal{X}$ ,

$$
\rho^{\pi}(x) \le u(x) := kg + 2 \cdot \min(2\mu, \max(x, g)).
$$

Notice that the upper bound  $u(x)$  is non-decreasing. Now, recall that  $\rho^*(x) = \min(h, \max(g, x))$  is also a non-decreasing mapping. Thus, policy π will α-pointwise approximate  $M$  for

$$
a = \max_{x \in X} \frac{u(x)}{\rho^*(x)}
$$

and since  $\mu \leq h$  and  $k = O(\log \frac{\mu}{g})$  this implies a  $O(\log \frac{\mu}{g} + \frac{\mu}{M})$ -pointwise approximation.

# <span id="page-25-0"></span>7 Pandora's Box with Optional Inspection and Semi-Local Approximation

Finally, we consider Pandora's Box with Optional Inspection (henceforth PBOI), a well-studied variant of Pandora's Box in which a box may be selected without first being opened. For brevity, we call selecting a box without first opening it "grabbing" the box. In the cost minimization (resp. reward maximization) setting, if the decision maker grabs a box, they pay (earn) the cost (value) inside of the box immediately, without ever having observed it. The availability of this extra action means that each box is now an MDP instead of a Markov chain, leading to a significantly more challenging optimization problem.

<span id="page-25-1"></span>Formally, we represent a box as a pair  $\mathcal{B} = (\mathcal{D}, c)$ , where  $\mathcal D$  is a distribution over nonnegative values, and  $c > 0$  is the opening cost. Figure [6](#page-25-1) illustrates the actions available to the algorithm.



Figure 6: A Pandora's box  $\mathcal{B} = (\mathcal{D}, c)$  under optional inspection is initially closed. In order to learn the value realization  $X \sim \mathcal{D}$ , the decision maker can open it (at a cost of c), after which they can select it to accept its value X. Alternatively, the decision maker can grab the box, meaning select it without opening it, receiving the box's value  $X$  without paying any cost. Because  $X$  is unknown whenever the box is being grabbed, for the purposes of expected value, receiving X is equivalent to receiving  $E[X] = m$ .

Previous work and our results. As we mentioned earlier, the maximization version of PBOI admits a PTAS in the single-item selection setting. However, prior to this work, there was no known policy that achieves better than a 1/2-approximation for matroid selection PBOI. Additionally, this 1/2-approximation is somewhat uninteresting. As [Beyhaghi and Kleinberg](#page-34-7) [\[2019\]](#page-34-7) observe, the optimal policy that does not grab any box will attain at least as much utility as the utility earned by the optimal algorithm from boxes that it opens, and the optimal policy that only grabs boxes will attain at least as much utility as the utility earned by the optimal algorithm from boxes it grabbed. Thus, the sum of these policy's utilities is an upper bound on the optimal algorithm and so uniformly randomizing between these policies will immediately give us a  $1/2$ -approximation.<sup>[8](#page-25-2)</sup>

One might wonder whether one can use local approximation to break the 1/2-approximation barrier in the matroid selection setting. Alas, in the maximization setting, we cannot: there exist boxes for which we cannot do better than a local  $(1/2 + \varepsilon)$ -approximation for any  $\varepsilon > 0$  (Theorem [7.2\)](#page-27-1). Our contribution is a novel approach to break the 1/2-approximation barrier in the matroid selection setting, proving the following result.

<span id="page-25-3"></span>**Theorem 7.1.** There exists an efficient randomized committing policy, namely Algorithm [3](#page-29-0) with appropriately chosen probabilities, that achieves a 0.582-approximation to the optimal adaptive policy for any instance of max-matroid-PBOI.

The key idea behind Theorem [7.1](#page-25-3) is a refinement of local approximation, which we call *semilocal ap*proximation (Definition [11\)](#page-28-1). Just as local approximations compose, we show in Theorem [7.3](#page-28-2) that semilocal approximations similarly compose. We then show in Theorem [7.7](#page-33-0) that, unlike ordinary local approximation,

<span id="page-25-2"></span><sup>8</sup>[Beyhaghi and Kleinberg](#page-34-7) [\[2019\]](#page-34-7) make this observation in the single-item setting, but combining it with results of [Singla](#page-35-0) [\[2017\]](#page-35-0) generalizes it to the matroid setting.

all boxes admit a semilocal approximation that leads to a 0.582-global approximation. Theorem [7.1](#page-25-3) then follows immediately from Theorems [7.3](#page-28-2) and [7.7.](#page-33-0)

There are two prices we pay in refining local approximation to semilocal approximation. First, the definition as we state it currently is specific to PBOI, though it could perhaps be generalized to other MDP families in the future. Second, semilocal composition only holds for matroid selection problems, whereas past works showed results compatible with any frugal (roughly, "greedy") algorithm [\[Singla, 2017,](#page-35-0) [Gupta et al.,](#page-35-2) [2019,](#page-35-2) [Scully and Doval, 2024\]](#page-35-1). We believe this second obstacle may be fundamental.

#### <span id="page-26-0"></span>7.1 Notation and preliminaries

In this section, we at times consider Pandora's box problems under both optional inspection, as illustrated in Figure [6,](#page-25-1) and mandatory inspection, which is the traditional setting [\[Weitzman, 1979\]](#page-35-8) that lacks the grab action. We use the same  $\mathcal{B} = (\mathcal{D}, c)$  notation for both, with context making it clear whether we are considering the box to be optional or mandatory inspection. We will apply our amortization framework for max-CICS, which is similar to the framework established in Section [3](#page-9-0) and Section [4,](#page-15-0) and is detailed in Appendix [E.](#page-49-0)

For a box  $\mathcal{B} = (\mathcal{D}, c)$  with optional inspection, the optimality curve of the local game  $(\mathcal{B}, y)$  is given by  $f_B(y) := \max\{E_{X\sim\mathcal{D}}[\max\{X, y\}] - c, E[X], y\}$ , with the quantities in the max representing the options of open, grab, or choosing the outside option, respectively. We can also write the optimality curve of the open action as  $f_{\mathcal{B}}^{\text{open}}(y) := \max\{E_{X\sim\mathcal{D}}[\max\{X,y\}]-c,y\}.$  These allow us to deduce the water draining surrogate values for the box. We define the following quantities of interest.

- The *mean value* is  $m = E_{X \sim \mathcal{D}}[X]$ .
- The Gittins index is g, the solution to  $c = \mathbb{E}_{X \sim \mathcal{D}}[(X g)^+]$ . Note that g is the water draining index of the policy that commits to opening the box, in other words, the largest outside option for which opening the box is better than accepting the outside option.
- The backup index is  $h = \max\{m, h'\}$ , where h' is the solution to  $c = \mathbb{E}_{X \sim \mathcal{D}}[(h' X)^+]$ . Note that h' is the largest outside option for which grabbing the box is better than opening it, and  $h$  is the largest outside option for which grabbing is better than both of the alternatives.

In terms of these quantities, we can rewrite the optimality curve for  $(\mathcal{B}, y)$  as:

$$
f_{\mathcal{B}}(y) = \begin{cases} m & \text{if } y < h \\ \mathbf{E} \left[ \max(y, \min(g, X)) \right] & \text{if } y \in [h, g] \\ y & \text{if } y > g \end{cases}
$$

Recall that the optimal surrogate value of the box, as well as the surrogate value for the opening action, are defined so that  $f_{\mathcal{B}}(y) = \mathbb{E}[\max\{W^*, y\}]$  and  $f_{\mathcal{B}}^{\text{open}}(y) = \mathbb{E}[\max\{W^{\text{open}}, y\}]$ . We therefore obtain:

- The opening surrogate value is  $W^{\text{open}} = \min\{X, g\}$ , where  $X \sim \mathcal{D}$  is the box's hidden value.
- The *optimal surrogate value* is  $W^* = \max{\min\{X, g\}, h\} = \max\{W^{\text{open}}, h\}$ , where  $X \sim \mathcal{D}$  is the box's hidden value.

When the box in question is not clear from context, we clarify using subscripts, e.g.  $g_1, g_2, \ldots$  for the Gittins indices of boxes  $\mathcal{B}_1, \mathcal{B}_2, \ldots$ . Finally, following along the lines of [Doval](#page-34-6) [\[2018,](#page-34-6) Assumption 1], we make an assumption for ease of presentation.

<span id="page-26-2"></span>**Assumption 1.** For all boxes  $\mathcal{B} = (\mathcal{D}, c)$  we consider, we assume the quantities defined above satisfy  $h \leq m$ . One can show that this further implies  $c \leq m \leq g$ . If a box  $\beta$  fails this condition, we replace it with a normalized box  $\mathcal{B}_{\text{norm}} = (m, 0)$ , which trivially satisfies  $h_{\text{norm}} = m_{\text{norm}} = g_{\text{norm}} = m$ .

This assumption says, roughly speaking, that every box has an open action that is sometimes worth taking. We emphasize that boxes which fail Assumption [1](#page-26-2) do not present an obstacle. In fact, quite the opposite: the grab action is the only worthwhile action for such boxes. See [Doval](#page-34-6) [\[2018,](#page-34-6) Section 2.2 and Appendix S.4] for

<span id="page-26-1"></span><sup>&</sup>lt;sup>9</sup>These follow from the following observations and plotting m, y, and  $E[\max\{X, y\}] - c$  as functions of y. (a)  $h = h'$  is the value of y such that  $m = \mathbb{E}_{X \sim \mathcal{D}}[\max\{X, y\}] - c$ . (b) g is the value of y such that  $y = \mathbb{E}_{X \sim \mathcal{D}}[\max\{X, y\}] - c$ . (c) As a function of y, the expression  $E_{X\sim\mathcal{D}}[\max\{X, y\}] - c$  has derivative in [0, 1], and at  $y = 0$ , its value is  $m - c$ .

further discussion. Rather than handling two cases throughout our presentation, we assume that all boxes are normalized in the way described in Assumption [1.](#page-26-2) Normalizing a box essentially removes the open action by replacing it with a "no-op". Crucially, normalizing a box does not change the optimal surrogate value  $W^*$ , which is always  $m$  for boxes that violate Assumption [1.](#page-26-2) As such, the bound we employ on the expected value of the optimal policy (Theorem [E.4\)](#page-51-1) remains true for the initial unnormalized system.

#### <span id="page-27-0"></span>7.2 Failure of Local Approximation for PBOI with Rewards

Committing policies for PBOI randomize over the two actions – open and grab – and can accordingly be described by specifying the probability of the open action, call it p. Recalling that the surrogate value for opening is  $W^{\text{open}}$  and the surrogate value for grabbing is m, we can adapt the notion of local approximation introduced in Definition [7](#page-15-1) for the PBOI setting as follows.[10](#page-27-2)

**Definition 10** (Local  $\alpha$ -approximation for PBOI). Consider a box  $\mathcal{B} = (\mathcal{D}, c)$ . We say the a probability  $p \in [0, 1]$  is a *local*  $\alpha$ -*approximation* for  $\beta$  if for all y,

$$
(1-p)E[\max\{W^{\text{open}}, y\}] + p \max\{m, y\} \ge E[\max\{\alpha W^*, y\}]
$$

Intuitively, this is saying that if we commit to grabbing the box with probability  $p$  and otherwise commit to inspecting it before opening it, we get a local  $\alpha$ -approximation. Unfortunately, this notion of local approximation only obtains a  $(1/2)$ -approximation in the worst case, as proven by the following lemma.

<span id="page-27-1"></span>**Theorem 7.2.** For all  $\alpha > 1/2$ , there exists a box that does not admit a local  $\alpha$ -approximation.

*Proof.* For any  $n \gg 1$ , consider the box  $\mathcal{B}^n$  where

$$
D = \begin{cases} 1 & \text{w.p. } 1 - \frac{1}{n^2} \\ n^3 & \text{w.p. } \frac{1}{n^2} \end{cases}
$$

and  $c = n - 1$ . Note that the mean of this distribution, m, is equal to  $n + 1 - \frac{1}{n^2}$ , and the Gittins index g is  $n^2$ . We get that  $\mathbb{E}[\max{\{\alpha W^*,0\}}]=\alpha m$  and  $\mathbb{E}[\max{\{W^{\text{open}},0\}}]=m-c$ . Similarly, if  $y=m$  then

$$
\mathbf{E}[\max\{\alpha W^*,y\}] = \max\{\mathbf{E}[\max\{\alpha W^{\text{open}},m\}],\alpha m\} = \mathbf{E}[\max\{\alpha W^{\text{open}},m\}]
$$

and then a straightforward calculation gets us that for all  $\alpha \geq \frac{1}{2}$  and sufficiently large n,

$$
\mathbf{E}[\max\{\alpha W^{\text{open}}, m\}] = \mathbf{E}_{X \sim \mathcal{D}}[\max\{\alpha X, m\}] - \alpha c = \alpha + m\left(1 - \frac{1}{n^2}\right).
$$

Thus, if we want to satisfy

$$
(1-p) \mathrm{E}[\max\{W^{\mathrm{open}},y\}]+p\max\{m,y\}\geq \mathrm{E}[\max\{\alpha W^*,y\}]
$$

at  $y = 0$ , we can plug in the quantities computed above and reorganize to get the bound  $p \ge \frac{m(\alpha-1)}{c} + 1$ . Similarly, we can plug in our computed quantities when  $y = m$  to get that when n is sufficiently large and  $\alpha \geq \frac{1}{2},$ 

$$
p\leq \frac{\alpha-1}{\frac{m}{n^2}-1}.
$$

Putting both of these bounds together, we get the following bound on  $\alpha$ ,

$$
\alpha \le \frac{m(c-m+n^2)}{mn^2 + cn^2 - m^2},
$$

which goes to  $\frac{1}{2}$  from above as  $n \to \infty$ . Therefore, for any  $\alpha > \frac{1}{2}$ , we can choose n large enough that  $\mathcal{B}^n$ does not admit a local  $\alpha$ -approximation.  $\Box$ 

<span id="page-27-2"></span><sup>10</sup>Also see [Scully and Doval](#page-35-1) [\[2024,](#page-35-1) Definition 4.2].

<span id="page-28-3"></span>To understand the behavior that leads to the worst case local  $\alpha$ -approximations, it will be illustrative to consider a box  $\mathcal{B}^n$  for large n as defined in the proof of Theorem [7.2.](#page-27-1) Figure [7](#page-28-3) plots the value of grabbing versus inspecting as a function of the outside value y for the box  $\mathcal{B}^{20}$ . As we can see, when y is small we have a strong preference for grabbing the box, and when  $y$  is larger we have a very slight preference to inspect the box. Looking at this plot makes it intuitively clear that the "always grab" policy performs quite well, since in the worst case there is only a small *additive* sub-optimality factor. However, local  $\alpha$ -approximation requires a small multiplicative sub-optimality factor, which, as the above proof demonstrates, the "always grab" policy does not achieve.



Figure 7: The red curve,  $\max\{m, y\}$ , represents the value from grabbing the box  $\mathcal{B}^{20}$  (as defined in the proof of Theorem [7.2\)](#page-27-1). The blue curve,  $E[\max\{W^{\text{open}}, y\}]$ , represents the value from inspecting the box  $\mathcal{B}^{20}$ . This box admits at most a local 0.537-approximation. By considering  $\mathcal{B}^n$  as  $n \to \infty$ , Theorem [7.2](#page-27-1) shows that for all  $\varepsilon > 0$ , there exist boxes admit at most a local  $(1/2 + \varepsilon)$ -approximation.

To capture this intuition, in Section [7.3,](#page-28-0) we introduce semilocal  $(\alpha, \beta)$ -approximation (Definition [11\)](#page-28-1), where  $\alpha$  captures the multiplicative suboptimality while  $\beta$  captures the additive suboptimality. We show that semilocal  $(\alpha, \beta)$ -approximations compose in the matroid selection setting leading to an overall  $(\alpha - \beta)$ approximation (Theorem [7.3\)](#page-28-2). Then, in Section [7.4,](#page-32-0) we show that all boxes admit a semilocal (0.682, 0.1) approximation (Theorem [7.7\)](#page-33-0), guaranteeing a 0.582-approximation in the matroid selection setting and thus breaking the existing barrier of 1/2.

#### <span id="page-28-0"></span>7.3 Semilocal Approximations and Their Composition

<span id="page-28-1"></span>**Definition 11** (Semilocal Approximation). Consider a box  $\mathcal{B} = (\mathcal{D}, c)$ . We say that a probability  $p \in [0, 1]$ is a semilocal  $(\alpha, \beta)$ -approximation for B if for all y,

<span id="page-28-4"></span>
$$
(1-p)E[\max\{W^{\text{open}},y\}] + p\max\{m,y\} \ge E[\max\{\alpha W^*,y\}] - p\beta m. \tag{3}
$$

If there exists a hedging probability that is a semilocal  $(\alpha, \beta)$ -approximation for the box  $\beta$ , then we say  $\beta$ admits semilocal  $(\alpha, \beta)$ -approximation.

Rewriting [\(3\)](#page-28-4) as

$$
(1-p)E[\max\{W^{\text{open}},y\}]+p(\max\{m,y\}+\beta m)\geq E[\max\{\alpha W^*,y\}].
$$

<span id="page-28-2"></span>makes it clear that semilocal  $(\alpha, \beta)$ -approximation is basically a local  $\alpha$ -approximation where we have boosted the value attained from grabbing by an additive factor proportional to the mean. It is important that we do this only for grabbing—boosting the value of inspecting by an additive factor would lead to trivial guarantees when we compose semilocal approximations in the matroid selection setting.

<span id="page-29-2"></span><span id="page-29-0"></span>Algorithm 3: Semilocal Approximation Composition Algorithm **Input:** A matroid PBOI instance  $\mathbb{I} = (\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{F})$  (normalized as described in Assumption [1\)](#page-26-2) A vector of probabilities  $(p_1, \ldots, p_n)$ 1 Relabel the boxes such that  $m_1 \geq \cdots \geq m_n$  $2 S^{\text{grab}} \leftarrow \{\}$ set of boxes marked as "grab" 3 For  $i \leftarrow 1, \ldots, n$ : 4  $L_i \leftarrow \textbf{if } S^{\text{grab}} \cup \{i\} \in \mathcal{F} \textbf{ then } 1 \textbf{ else } 0$  $L_i = 0$  means we never want to grab box i 5 Sample  $K_i \leftarrow \text{Bernoulli}(p_i)$   $K_i = 1$  means provisionally mark box i "grab" 6 If  $K_i = 1$  and  $L_i = 1$ :  $\begin{array}{c|c} \mathbf{7} & \end{array} \begin{array}{c} \end{array} \begin{array}{c} \mathbf{7} \\ \mathbf{8} \end{array} \begin{array}{c} \mathbf{8} \\ \mathbf{9} \end{array} \begin{array}{c} \mathbf{9} \\ \mathbf{10} \end{array} \begin{array}{c} \mathbf{9} \\ \mathbf{10} \end{array}$ fully mark box  $i$  as "grab"  $\bf{s}$  Mark the boxes in  $S^{\rm{grab}}$  as "grab" and the rest as "open"

<span id="page-29-1"></span><sup>9</sup> Run the frugal Gittins policy [\[Singla, 2017\]](#page-35-0) on the resulting mandatory-inspection instance

**Theorem 7.3** (Composition of Semilocal Approximations). Let  $\alpha > \beta \geq 0$ , let  $\mathbb{I} = (\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{F})$  be a max-matroid-CICS instance, and let  $(p_1, \ldots, p_n)$  be a vector of probabilities such that  $p_i$  is a semilocal  $(\alpha, \beta)$ -approximation for  $\mathcal{B}_i$ . Then there exists a committing policy, namely Algorithm [3,](#page-29-0) that is an  $(\alpha - \beta)$ approximation for the CICS instance I.

Proof. There are two main steps of the proof, each stated and proved in a lemma below. We express both steps in terms of the random variables

$$
W_i^{\text{alg}} = \begin{cases} W_i^{\text{open}} & \text{if } K_i L_i = 0\\ m_i & \text{if } K_i L_i = 1, \end{cases}
$$

where  $K_i$  and  $L_i$  are as defined in Algorithm [3.](#page-29-0) One can think of  $W_i^{\text{alg}}$  as the surrogate value of box i conditional on the state of the algorithm at Line [7.](#page-29-1)

- The first step, Lemma [7.4,](#page-30-0) is to express the value achieved by Algorithm [3](#page-29-0) in terms of  $W_i^{\text{alg}}$ .
- The second step, Lemma [7.5,](#page-30-1) is to compare the resulting expression to an upper bound on the optimal value. This step uses the semilocal  $(\alpha, \beta)$ -approximation guarantee from Definition [11,](#page-28-1) which gives us a relationship between  $W_i^{\text{alg}}$  and  $W_i^*$ .

Combining the lemmas yields

$$
\mathbb{E}[\text{value achieved by Algorithm 3}] \geq \mathbb{E}\bigg[\max_{S \in \mathcal{F}} \sum_{i \in S} \alpha W_i^* - \sum_{i \in S^{\text{graph}}} \beta m_i\bigg],
$$

where  $S<sup>graph</sup>$  is the set of boxes marked "grab" at Line [7,](#page-29-1) i.e. at the end of the algorithm. It remains only to relate the two terms on the right-hand side to the optimal expected value.

• Theorem [E.4](#page-51-1) implies

$$
\mathbf{E}\left[\max_{S \in \mathcal{F}} \sum_{i \in S} \alpha W_i^*\right] \ge \alpha \mathbf{E}[\text{value of optimal policy}].
$$

• Because Algorithm [3](#page-29-0) ensures  $S^{\text{graph}} \in \mathcal{F}$  by construction, the following policy is feasible: "Compute  $S<sup>graph</sup>$  as in Algorithm [3,](#page-29-0) but then simply grab the boxes in  $S<sup>graph</sup>$ ." This algorithm achieves value  $\mathop{\mathrm{E}}\bigl[\sum_{i \in S^{\mathrm{grad}}} m_i\bigr]$ , which means

$$
\mathbf{E}\bigg[\sum_{i\in S^{\text{grab}}} \beta m_i\bigg] \leq \beta \mathbf{E}[\text{value of optimal policy}].
$$

Therefore, as desired, E[value achieved by Algorithm [3](#page-29-0)]  $\geq (\alpha - \beta)E$ [value of optimal policy].

<span id="page-30-0"></span>**Lemma 7.4.** For any max-matroid Pandora's box instance  $\mathbb{I} = (\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{F})$  and any vector of probabilities  $(p_1, \ldots, p_n)$ , the expected value achieved by Algorithm [3](#page-29-0) is

$$
\text{E}[\text{value achieved by Algorithm 3}] = \text{E} \bigg[ \max_{S \in \mathcal{F}} \sum_{i \in S} W^\text{alg}_i \bigg]
$$

Proof. Consider running Algorithm [3](#page-29-0) through Line [7,](#page-29-1) but not further. All of the randomness thus far comes from the coin flips  $K_i$ , and no boxes have been opened yet. This means that conditional on the coin flips  $K_i$ , the expected value achieved is that of a mandatory-inspection instance  $\mathbb{I}' = (\mathcal{B}'_1, \ldots, \mathcal{B}'_n, \mathcal{F})$  whose ith box  $\mathcal{B}'_i$  is defined as follows:

• If  $K_i L_i = 0$  (marked "open"),  $\mathcal{B}'_i = (\mathcal{D}_i, c_i)$ , i.e. the box is the same as the original instance.

• If  $K_i L_i = 1$  (marked "grab"),  $\mathcal{B}'_i = (m_i, 0)$ , i.e. the box is free to open and always contains value  $m_i$ . Under instance  $\mathbb{I}'$ , box *i*'s surrogate value is given by  $W_i^{\text{alg}}$ , so

$$
\text{E}[\text{value achieved by Algorithm 3 } | K_1, \dots, K_n] = \text{E} \left[ \max_{S \in \mathcal{F}} \sum_{i \in S} W_i^{\text{alg}} \middle| K_1, \dots, K_n \right]
$$

The result then follows by the law of total expectation.

<span id="page-30-1"></span>Lemma 7.5. Under the hypotheses of Theorem [7.3,](#page-28-2)

<span id="page-30-2"></span>
$$
\mathcal{E}\left[\max_{S \in \mathcal{F}} \sum_{i \in S} W_i^{\text{alg}}\right] \ge \mathcal{E}\left[\max_{S \in \mathcal{F}} \sum_{i \in S} \alpha W_i^* - \sum_{i \in S^{\text{grab}}} \beta m_i\right],\tag{4}
$$

.

where  $S<sup>graph</sup>$  refers to the value of the  $S<sup>graph</sup>$  variable from Algorithm [3](#page-29-0) at Line [7.](#page-29-1)

*Proof.* The outline of the proof is as follows. We begin with the left-hand side of  $(4)$ . Then, for each box i, we swap  $W_i^{\text{alg}}$  with  $\alpha W_i^*$ , and also subtract  $\beta m_i$  if box i is marked "grab". Because each box admits a semilocal  $(\alpha, \beta)$ -approximation, each of these replacements only decreases the expression's expected value. After all n replacements, we are left with the right-hand side of [\(4\)](#page-30-2), as desired. This is the same strategy used by [Scully](#page-35-1) [and Doval](#page-35-1) [\[2024,](#page-35-1) Theorem 5.4], but some careful conditioning is needed to account for the  $\beta m_i$  subtractions.

In order to notate the one-by-one replacement outlined above, let

$$
W_i^{(j)} = \begin{cases} W^{\text{alg}} & \text{if } i \leq j \\ \alpha W^* & \text{if } i > j, \end{cases} \hspace{2cm} U^{(j)} = \max_{S \in \mathcal{F}} \sum_{i \in S} W_i^{(j)}.
$$

Using this notation and recalling how  $S<sup>grab</sup>$  is defined in Algorithm [3,](#page-29-0) we can rewrite our goal [\(4\)](#page-30-2) as

$$
\mathcal{E}[U^{(n)}] \ge \mathcal{E}\bigg[U^{(0)} - \sum_{i=1}^{n} \beta K_i L_i m_i\bigg].
$$

Therefore, it suffices to show that for all  $j \in \{1, \ldots, m\}$ ,

<span id="page-30-3"></span>
$$
\mathcal{E}[U^{(j)}] \ge \mathcal{E}\bigg[U^{(j-1)} - \beta K_j L_j m_j\bigg].\tag{5}
$$

We will show [\(5\)](#page-30-3) using Definition [11.](#page-28-1) But in order to do so, we need to express each side in terms of a maximum between  $W_j^{\text{alg}}$  or  $\alpha W_j^*$  and a quantity that is independent of box j's value  $X_j$  and coin flip  $K_j$ . We express the latter quantity in terms of

$$
Y_{\neq j} = \max_{S \in \mathcal{F}: j \notin S} \sum_{i \in S} W_i^{(j)}, \qquad Z_{\neq j} = \max_{S \in \mathcal{F}: j \in S} \sum_{i \in S \setminus \{j\}} W_i^{(j)}.
$$

 $\Box$ 

.

$$
\Box
$$

These can both be seen as optimal total surrogate values achievable without box j. The difference is that  $Y_{\neq j}$  optimizes over sets that exclude j, whereas  $Z_{\neq j}$  optimizes over sets that include j (but still excludes box  $j$ 's surrogate value from the sum).

With the definitions of  $Y_{\neq j}$  and  $Z_{\neq j}$  in hand, we can express  $U^{(j)}$  and  $U^{(j-1)}$  as

$$
U^{(j)} = \max\{Y_{\neq j}, Z_{\neq j} + W_j^{\text{alg}}\} = Z_{\neq j} + \max\{W_j^{\text{alg}}, Y_{\neq j} - Z_{\neq j}\},
$$
  

$$
U^{(j-1)} = \max\{Y_{\neq j}, Z_{\neq j} + \alpha W_j^*\} = Z_{\neq j} + \max\{\alpha W_j^*, Y_{\neq j} - Z_{\neq j}\}.
$$

So to show [\(5\)](#page-30-3), it suffices to show

$$
\mathbb{E}[\max\{W_j^{\text{alg}}, Y_{\neq j} - Z_{\neq j}\}] \le \mathbb{E}[\max\{W_j^{\text{alg}}, Y_{\neq j} - Z_{\neq j}\} - \beta K_j L_j m_j].
$$

Letting  $K_{\leq j} = (K_1, \ldots, K_{j-1})$ , by the law of total expectation, it suffices to show

<span id="page-31-0"></span>
$$
\mathbb{E}[\max\{W_j^{\text{alg}}, Y_{\neq j} - Z_{\neq j}\} \mid K_{ (6)
$$

The key to showing [\(6\)](#page-31-0) is observing the following independence facts:

- $(K_j, X_j)$  is independent of  $K_{\leq j}$ . This is because the coin flips  $K_{\leq j}$  affect neither the coin flip  $K_j$  nor the box value  $X_j$ .
- $(K_j, X_j)$  is conditionally independent of  $(Y_{\neq j}, Z_{\neq j})$  given  $K_{\leq j}$ . This is because once  $K_{\leq j}$  are fixed, the values  $(Y_{\neq j}, Z_{\neq j})$  are a function of the values of boxes other than j, and the box values are mutually independent.

The main obstacle to applying Definition [11](#page-28-1) to [\(6\)](#page-31-0) is that  $W_j^{\text{alg}}$  depends on  $L_j$ , which in turn depends on  $K_{\leq i}$ . Fortunately, we see from Algorithm [3](#page-29-0) that

$$
L_{\leq j} = (L_1, \ldots, L_j)
$$
 is a deterministic function of  $K_{< j} = (K_1, \ldots, K_{j-1}).$ 

This is because for all  $i$ , when executing Line [3,](#page-29-2) the only randomness the algorithm has used is the past coin flips  $K_{\leq i}$ . So to show [\(6\)](#page-31-0), we split into cases based on whether  $L_i = 0$  or  $L_i = 1$ .

Suppose that  $L_j = 1$ . More precisely, suppose  $K_{< j} = k_{< j}$ , where  $k_{< j}$  is any bit vector such that  $K_{< j} = k_{< j}$ induces  $L_j = 1$  in Algorithm [3.](#page-29-0) In this case, the coin flip  $K_j$  impacts whether we mark box j as "grab" or """" open", so we will use the semilocal approximation guarantee from Definition [11.](#page-28-1) By the assumption on  $k_{< j}$ and the fact that  $K_j \sim \text{Bernoulli}(p_j)$  independently of  $K_{\leq j}$ ,

$$
E[K_j L_j \mid K_{
$$

So, by Definition [11,](#page-28-1)

$$
\begin{aligned} \mathbf{E}[\max\{W_j^{\text{alg}}, y\} \mid K_{
$$

The fact that  $(Y_{\neq j}, Z_{\neq j})$  is conditionally independent of  $(K_j, X_j)$  given  $K_{< j}$  completes the proof of [\(6\)](#page-31-0) on the event  $L_j = 1$ .

Suppose now that  $L_j = 0$ . In this case, we mark box j as "open" regardless of the coin flip  $K_j$ , so instead of using Definition [11,](#page-28-1) we will show that marking box  $j$  as "open" does not lose any any potential value. Specifically, we will show the following:

- (a) For all  $y \ge h_j$ , we have  $E[\max\{W_j^{\text{open}}, y\}] = E[\max\{W_j^*, y\}].$
- (b) If  $L_j = 0$ , then  $Y_{\neq j} Z_{\neq j} \geq m_j$ .

Together with the fact that  $m_j \ge h_j$  (Assumption [1\)](#page-26-2), facts (a) and (b) imply that for any  $k_{\le j}$  such that  $K_{\leq j} = k_{\leq j}$  induces  $L_j = 0$  in Algorithm [3,](#page-29-0)

$$
E[\max\{W_j^{\text{alg}}, Y_{\neq j} - Z_{\neq j}\} | K_{  
\n
$$
= E[\max\{W_j^*, Y_{\neq j} - Z_{\neq j}\} | K_{  
\n
$$
\geq E[\max\{\alpha W_j^*, Y_{\neq j} - Z_{\neq j}\} | K_{  
\n
$$
= E[\max\{\alpha W_j^*, Y_{\neq j} - Z_{\neq j}\} - \beta K_j L_j m_j | K_{
$$
$$
$$
$$

which completes the proof of [\(6\)](#page-31-0) on the event  $L<sub>j</sub> = 0$ . It remains only to show (a) and (b). For (a), observe that if  $y \ge h_j$ , then  $y_j = \max\{h_j, y_j\}$ , so

$$
\mathbf{E}[\max\{W^{\mathrm{open}}_j,y\}]=\mathbf{E}[\max\{W^{\mathrm{open}}_j,h_j,y\}]=\mathbf{E}[\max\{W^*_j,y\}].
$$

For (b), we will show that if  $L_j = 0$ , then  $Y_{\neq j} \ge m_j + Z_{\neq j}$ . Let

- $B^Z \in \text{argmax}_{S \in \mathcal{F}: j \in S} \sum_{i \in S \setminus \{j\}} W_i^{(j)}$  be a maximizing basis in the definition of  $Z_{\neq j}$ ,
- $S_{\leq j}^{\text{graph}} = S^{\text{graph}} \cap \{1, \ldots, j-1\}$  be the boxes marked "grab" before the  $i = j$  for loop iteration, and
- $B^{\text{grab}}$  be  $S_{\leq j}^{\text{grab}}$  extended to a basis by elements of  $B^Z$ , so that  $S_{\leq j}^{\text{grab}} \subseteq B^{\text{grab}} \subseteq S_{\leq j}^{\text{grab}} \cup B^Z$ .

Because  $L_j = 0$ , we have  $S_{\leq j}^{\text{grab}} \cup \{j\} \notin \mathcal{F}$ , which means  $j \notin B^{\text{grab}}$ . But  $j \in B^Z$  by definition, so  $j \in B^Z \setminus B^{\text{grab}}$ . By the basis exchange property, there exists  $k \in B^{\text{grab}} \setminus B^Z$  such that the following is a basis:

$$
B^Y = (B^Z \setminus \{j\}) \cup \{k\}.
$$

But  $B^{\text{grab}} \setminus B^Z \subseteq S^{\text{grab}}_{\leq j}$ , which means  $W_k^{(j)} = m_k \geq m_j$ . This means

$$
Y_{\neq j} \ge \sum_{i \in B_Y} W_i^{(j)} = m_k + \sum_{i \in B_Z \setminus \{j\}} W_i^{(j)} \ge m_j + \sum_{i \in B_Z \setminus \{j\}} W_i^{(j)} = m_j + Z_{\neq j}.
$$

#### <span id="page-32-0"></span>7.4 Breaking the 1/2 Barrier with Semilocal Approximation

<span id="page-32-1"></span>**Lemma 7.6.** Consider a box B satisfying Assumption [1.](#page-26-2) For all  $\beta \in \mathbb{R}$  there exists a  $p \in [0,1]$  that is a semilocal  $(\alpha(\beta), \beta)$ -approximation for B where

$$
\alpha(\beta) = \begin{cases} \frac{1}{1 + \frac{c}{m} - \beta \frac{c}{m - c}} & \text{if } 1 \ge \frac{1}{1 + \frac{c}{m} - \beta \frac{c}{m - c}} > 0\\ 1 & \text{otherwise.} \end{cases}
$$

Moreover, the p that achieves this is  $p = \alpha(\beta) \frac{c}{m}$ .

*Proof.* To simplify notation we denote  $\alpha(\beta)$  as  $\alpha$  throughout the proof. Proving this lemma amounts to showing that this  $\alpha$  and  $\beta$  satisfy [\(3\)](#page-28-4),

$$
(1-p) \mathrm{E}[\max\{W^{\mathrm{open}},y\}]+p\max\{m,y\}\geq \mathrm{E}[\max\{\alpha W^*,y\}]-p\beta m
$$

for all  $y \geq 0$ . This is equivalent to showing that

$$
f(y)=(1-p)\mathrm{E}[\max\{W^{\mathrm{open}},y\}]+p\max\{m,y\}-\mathrm{E}[\max\{\alpha W^*,y\}]+p\beta m
$$

satisfies  $f(y) \ge 0$  for all  $y \ge 0$ . We will first show that it is sufficient to check that  $f(0) \ge 0$  and that  $f(y) \ge 0$ for  $y \geq m$ . It follows from the definition of h that at all values y for which  $f'(y)$  is defined, it is equal to

$$
(1-p)\Pr\left[y \geq W^{\text{open}}\right] + pI(y > m) - I(y > \alpha h)\Pr\left[y > \alpha W^{\text{open}}\right].
$$

Observe that

• if  $y < \alpha h < m$ , then  $f'(y) \geq 0$ ,

• if  $\alpha h < y < m$ , then since  $\alpha \leq 1$ , we must have  $f'(y) \leq 0$ .

Thus a global minimum of  $f(y)$  must be at  $y = 0$  or on  $y \ge m$ , so it is sufficient to check that [\(3\)](#page-28-4) is satisfied when  $y = 0$  and  $y \ge m$ . When  $y = 0$ , [\(3\)](#page-28-4) reduces to

$$
(1 - p)(m - c) + pm \geq \alpha m - p\beta m.
$$

which holds for the values of  $\alpha$  and p given in the lemma. When  $y \geq m$ , [\(3\)](#page-28-4) reduces to

<span id="page-33-1"></span>
$$
(1-p)E[\max\{W^{\text{open}},y\}] + py \ge E[\max\{\alpha W^*,y\}] - p\beta m = \alpha E[\max\{W^{\text{open}},\frac{y}{\alpha}\}] - p\beta m \tag{7}
$$

where the last equality follows from the fact that  $E[\max{\{\alpha W^*, y\}}] = \max{E[\max{\{\alpha W^{\text{open}}, y\}}], \alpha m}$ . The slope of  $E[\max\{W^{\text{open}}, y\}]$  is bounded above by 1, so

$$
\alpha \mathbb{E}[\max\{W^{\text{open}}, \frac{y}{\alpha}\}] \leq \alpha \mathbb{E}[\max\{W^{\text{open}}, y\}] + \alpha \left(\frac{y}{\alpha} - y\right).
$$

Applying this bound to [\(7\)](#page-33-1), we find that [\(7\)](#page-33-1) holds if,

<span id="page-33-2"></span>
$$
(\alpha + p - 1)(\mathbb{E}[\max\{W^{\text{open}}, y\}] - y) \le p\beta m. \tag{8}
$$

Since the slope of  $E[\max\{W^{\text{open}}, y\}]$  is bounded above by 1, it must be that  $E[\max\{W^{\text{open}}, y\}] - y$  is maximized at  $y = 0$  where it is equal to  $m - c$ , so [\(8\)](#page-33-2) holds if,

$$
(\alpha + p - 1)(m - c) \leq p\beta m.
$$

Observe that if  $p = \alpha \frac{c}{m}$ , we can rewrite this inequality as

$$
\alpha \left( 1 + \frac{c}{m} - \frac{c}{m - c} \beta \right) \le 1,
$$

which is satisfied by the  $\alpha$  given in the lemma.

<span id="page-33-0"></span>**Theorem 7.7.** For any box B satisfying Assumption [1,](#page-26-2) if  $\beta = \frac{1}{10}$ , the p given in Lemma [7.6](#page-32-1) is a semilocal  $(\alpha, \beta)$ -approximation for B with  $\alpha \geq 0.682$ .

*Proof.* Lemma [7.6](#page-32-1) guarantees that for  $\beta = \frac{1}{10}$ , there is some  $p \in [0, 1]$  that is a semilocal  $(\alpha, \beta)$ -approximation for  $\mathcal B$  with

$$
\alpha = \begin{cases} \frac{1}{1 + \frac{c}{m} - \frac{c}{10(m-c)}} & \text{if } 1 \ge \frac{1}{1 + \frac{c}{m} - \frac{c}{10(m-c)}} > 0\\ 1 & \text{otherwise.} \end{cases}
$$

It is straightforward to verify (using a computer algebra system or numerical solver) that since  $\frac{c}{m} \in [0,1]$ ,  $1/(1+\frac{c}{m}-\frac{c}{10(m-c)}) \ge 0.682$  when  $1 \ge 1/(1+\frac{c}{m}-\frac{c}{10(m-c)}) > 0$ .

 $\Box$ 

# <span id="page-34-0"></span>References

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# Appendix

# <span id="page-36-0"></span>A Omitted Proofs from Section [3](#page-9-0)

In this chapter of the appendix, we present all omitted proofs from Section [3.](#page-9-0) We restate all our results for the reader's convenience.

#### <span id="page-36-1"></span>A.1 Amortized Surrogate Costs for Markov Chains

In this section, we provide the formal proofs and extra details on all our results for matroid-min-CICS over Markov chains that we established in Section [3.1.](#page-9-1)

Amortization Lower Bound. We begin with the proof of the lower bound we stated in Lemma [3.1,](#page-9-2) relating the cost of any algorithm to the surrogate costs of any amortization. We note that one way to prove Lemma [3.1](#page-9-2) is to follow the approach developed by [Singla](#page-35-0) [\[2017\]](#page-35-0) and relate the performance of any algorithm on a "costly information" instance to its performance in a "free information world" where action costs are paid by an outside investor who is in turn paid back the extra amortized cost of any terminal state that the algorithm accepts. Instead, we provide an algorithmic proof that will allow us to directly argue about the optimality of the water filling index policy, as well as extend our setting to MDPs.

**Lemma 3.1.** Consider a matroid-min-CICS  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  over Markov chains and let  $b_i$  be any cost amortization of  $\mathcal{M}_i$  with surrogate cost  $\rho_i := \rho_{\mathcal{M}_i,b_i}$  for all  $i \in [n]$ . Then, the expected cost of any algorithm for  $\mathbb{I}$  is at least  $E \left[ \min_{S \in \mathcal{F}} \sum_{i \in S} \rho_i \right]$ .

Proof. Let  $\mathcal{M}_i = (S_i, \sigma_i, A_i, c_i, \mathcal{D}_i, V_i, T_i)$  for  $i \in [n]$  be the Markov chains comprising I and let ALG be any algorithm for I. We define the following random events:

- $I(s_i) :=$  the event that ALG advances  $\mathcal{M}_i$  at the non-terminal state  $s_i \in S_i$ .
- $A(t_i) :=$  the event that ALG accepts the terminal state  $t_i \in T_i$ .
- $X(i) :=$  the event that ALG accepts a terminal state of  $\mathcal{M}_i$ .

Then, we can write the expected cost of the algorithm as

$$
E\left[\text{cost}(\text{ALG})\right] = \sum_{i=1}^{n} \Bigg( \sum_{s_i \in S_i} \Pr\left[I(s_i)\right] \cdot c_i(s_i) + \sum_{t_i \in T_i} \Pr\left[A(t_i)\right] \cdot v_i(t_i) \Bigg)
$$
  
\n
$$
= \sum_{i=1}^{n} \Bigg( \sum_{s_i \in S_i} \Pr\left[I(s_i)\right] \cdot c_i(s_i) + \sum_{t_i \in T_i} \Pr\left[A(t_i)\right] \cdot \left(\rho_{b_i}(t_i) - \sum_{s_i: t_i \in T(s_i)} b_{s_i t_i}\right) \Bigg)
$$
  
\n
$$
= \sum_{i=1}^{n} \sum_{t_i \in T_i} \Pr\left[A(t_i)\right] \cdot \rho_{b_i}(t_i) + \sum_{i=1}^{n} \sum_{s_i \in S_i} \Bigg( \Pr\left[I(s_i)\right] \cdot c_i(s_i) - \sum_{t_i \in T(s_i)} \Pr\left[A(t_i)\right] \cdot b_{s_i t_i} \Bigg)
$$

We will now show the following inequalities:

<span id="page-36-2"></span>
$$
\sum_{i=1}^{n} \sum_{t_i \in T_i} \Pr\left[A(t_i)\right] \cdot \rho_{b_i}(t_i) \ge \mathcal{E}\left[\min_{S \in \mathcal{F}} \sum_{i \in S} \rho_i\right]
$$
\n(9)

<span id="page-36-3"></span>
$$
\Pr\left[I(s_i)\right] \cdot c_i(s_i) \ge \sum_{t_i \in T(s_i)} \Pr\left[A(t_i)\right] \cdot b_{s_i t_i} \quad \forall i \in [n], s_i \in S_i \tag{10}
$$

that will directly imply the lemma.

We begin with inequality [9.](#page-36-2) A necessary condition for the algorithm to accept a terminal state  $t_i \in T_i$  is for this terminal to be realized by the Markov chain; in general, we use  $p_i(s_i)$  to denote the probability that a random walk on chain  $\mathcal{M}_i$  passes through state  $s_i \in \mathcal{S}_i$  and  $\mathcal{Q}(s_i)$  to denote the corresponding event. Then, for  $A(t_i)$  to happen a necessary condition is for  $Q(t_i)$  to happen. Furthermore, recall that by definition the surrogate cost  $\rho_i$  corresponds to sampling a terminal state  $t_i$  with probability  $p_i(t_i)$  and returning  $\rho_{b_i}(t_i)$ . Thus:

$$
\sum_{i=1}^{n} \sum_{t_i \in T_i} \Pr\left[A(t_i)\right] \cdot \rho_{b_i}(t_i) = \sum_{i=1}^{n} \sum_{t_i \in T_i} \Pr\left[A(t_i)|Q(t_i)\right] \cdot p_i(t_i) \cdot \rho_{b_i}(t_i)
$$
\n
$$
= \sum_{i=1}^{n} \sum_{t_i \in T_i} p_i(t_i) \cdot E\left[X(i) \cdot \rho_i|Q(t_i)\right]
$$
\n
$$
= \sum_{i=1}^{n} E\left[X(i) \cdot \rho_i\right]
$$
\n
$$
\geq E\left[\min_{S \in \mathcal{F}} \sum_{i \in S} \rho_i\right]
$$

where the last inequality follows from feasibility; the set of Markov chains selected by the algorithm (i.e. the set  $S = \{i : X(i) = 1\}$  must be feasible, and thus  $S \in \mathcal{F}$ .

We now proceed to inequality [10.](#page-36-3) The proof follows from the crucial fact that for each state  $s_i \in S_i$  and each terminal  $t_i \in T_i(s_i)$ , it holds that  $Pr[A(t_i)|Q(t_i)] \leq Pr[I(s_i)|Q(s_i)]$ . To see why this is true, observe that for  $A(t_i)$  to happen both  $Q(t_i)$  (the algorithm cannot accept a non-realized terminal state) and  $I(s_i)$ (the algorithm needs to advance all the states in the unique path from  $\sigma_i$  to  $t_i$  in order to accept  $t_i$ ) need to happen. Likewise, for  $I(s_i)$  to happen it must be the case that  $Q(s_i)$  happened. Chaining these events, we have

$$
\Pr\left[A(t_i)|Q(t_i)\right] = \frac{\Pr\left[A(t_i)\right]}{p_i(t_i)} = \frac{\Pr\left[A(t_i)|I(s_i)\right] \cdot \Pr\left[I(s_i)|Q(s_i)\right] \cdot p_i(s_i)}{p_i(t_i)} = \Pr\left[I(s_i)|Q(s_i)\right] \cdot \frac{\Pr\left[A(t_i)|I(s_i)\right]}{\Pr\left[Q(t_i)|Q(s_i)\right]}
$$

and the claim follows by  $Pr[A(t_i)|I(s_i)] \leq Pr[Q(t_i)|Q(s_i)]$ ; this is true since even conditioned on advancing  $s_i$ , in order to accept  $t_i$  is still needs to be realized and this happens with probability  $Pr[Q(t_i)|Q(s_i)]$ . From this, we finally have that

$$
\sum_{t_i \in T(s_i)} \Pr\left[A(t_i)\right] \cdot b_{s_i t_i} = \sum_{t_i \in T(s_i)} \Pr\left[A(t_i)|Q(t_i)\right] \cdot p_i(t_i) \cdot b_{s_i t_i}
$$
\n
$$
\leq \sum_{t_i \in T(s_i)} \Pr\left[I(s_i)|Q(s_i)\right] \cdot p_i(t_i) \cdot b_{s_i t_i}
$$
\n
$$
= \Pr\left[I(s_i)|Q(s_i)\right] \cdot \sum_{t_i \in T(s_i)} p_i(t_i) \cdot b_{s_i t_i}
$$
\n
$$
= \Pr\left[I(s_i)|Q(s_i)\right] \cdot c_i(s_i) \cdot p_i(s_i)
$$
\n
$$
= \Pr\left[I(s_i)\right] \cdot c_i(s_i).
$$

where the fourth line follows by definition of amortization. This completes the proof.

 $\Box$ 

<span id="page-37-0"></span>Observe that in the proof of Lemma [3.1,](#page-9-2) we only used inequalities at two points: (i) when relating the amortized cost of the terminals that the algorithm selected to the set of terminals of minimum realized surrogate cost and (ii) when we established that  $Pr[A(t_i)|I(t_i)] \leq Pr[Q(t_i)|Q(s_i)]$ ; notice that this was only required for pairs  $(s_i, t_i)$  with  $b_{s_i t_i} > 0$ . The first inequality can become tight if the algorithm somehow ensures that it will always accept the feasible set of realized terminal states whose total surrogate cost is minimum. The second inequality becomes tight if the algorithm can somehow ensure that whenever it advances a state s sharing cost  $b_{st} > 0$  with one of its terminal states  $t \in T(s)$ , then the algorithm will always accept  $t$  conditioned on it being realized. This allows us to characterize the conditions under which the lower bound is actually met by an algorithm, which we formally state as the following corollary:

**Corollary A.1.** Consider a matroid-min-CICS  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  over Markov chains and let  $b_i$  be any cost amortization of  $\mathcal{M}_i$  with surrogate cost  $\rho_i := \rho_{\mathcal{M}_i,b_i}$  for all  $i \in [n]$ . Then, any algorithm that satisfies both

- 1. Surrogate Optimality. The algorithm always accepts the feasible set of terminal states of minimum total surrogate cost.
- 2. **Promise of Payment.** Whenever one of the Markov chains  $\mathcal{M}_i$  gets advanced to a state  $s_i \in S_i$  such that  $b_{s_i't_i} > 0$  for some terminal state  $t_i \in T(s_i)$  and some ancestor state  $s_i'$  of  $s_i$ , the algorithm will immediately advance  $s_i$ .

will have expected cost precisely  $E\left[\min_{S \in \mathcal{F}} \sum_{i \in S} \rho_i\right]$ .

We note the the promise of payment property states that whenever nature realizes a terminal state that shares a fraction of the cost of some of the actions leading to it, and the algorithm takes one of these actions, then it will ensure that this state is accepted. This ensures that the algorithm will pay back (in expectation) the promised cost shares of all the actions it takes.

Water Filling Amortization. We proceed to algorithmically define water filling amortization. Fix any Markov chain  $\mathcal{M} = (S, \sigma, A, c, \mathcal{D}, V, T)$  and for each state s, let lev(s) denote its level; that is, the maximum length of a path from s to some terminal state in  $T(s)$ . By definition, the level of all terminal states is 0 and the level of the starting state  $\sigma$  equals the horizon H of the Markov chain. The algorithmic construction of water filling amortization is given in Algorithm [4.](#page-38-0)

Algorithm 4: Water Filling Amortization

**Input:** A Markov chain  $\mathcal{M} = (S, \sigma, A, c, \mathcal{D}, V, T)$ . 1 For each terminal state t, initialize  $w(t) := v(t)$ . 2 for  $\ell = 1$  to H do 3 for each state  $s \in S$  with  $\text{lev}(s) = \ell$  do 4 Order  $T(s)$  by increasing current costs:  $w(t_1) \leq w(t_2) \leq \cdots \leq w(t_m)$ . 5 Set  $b_{st} := 0$  for all  $t \in T(s)$ . 6 | Initialize  $i := 0$  and let  $w(t_{m+1}) := \infty$ .  $\begin{array}{ll} \pi & \mid & \mid & \text{while } c(s)\cdot p(s) > \sum_{t\in T(s)} p(t)\cdot b_{st} \textbf{ do} \end{array}$  $\bullet$  | | Increment  $i := i + 1$ . 9 | | Compute threshold  $T := (c(s)p(s) - \sum_{t \in T(s)} p(t)b_{st})/(\sum_{j \leq i} p(t_j)).$ 10 Set  $b_{st_i} := \min\{T, w(t_{i+1}) - w(t_i)\}\$ and  $w(t_i) := w(t_i) + b_{st_i}$ . 11 | | For all  $j < i$ , update  $b_{st_j} := b_{st_j} + b_{st_i}$  and  $w(t_j) := w(t_j) + b_{st_i}$ .  $12$  end <sup>13</sup> end <sup>14</sup> end **Output:** The water filling amortization  $\{b_{st}\}\$ for all  $s \in S$  and  $t \in T(s)$ .

<span id="page-38-0"></span>The algorithm visits the states of the Markov chain in increasing level order. When considering state s, it needs to determine cost shares  $b_{st} \geq 0$  such that  $c(s)p(s) = \sum_{t \in T(s)} p(t)b_{st}$ . It starts by distributing its cost to the terminal state  $t_1$  of minimum current cost  $w(t_1)$  until it matches the current cost of the terminal state  $t_2$  that has the second minimum cost; after that point, the two terminals get grouped together and are increased simultaneously from now on. Computing the threshold  $T$  at every iteration ensures that we will never allocate more cost than necessary.

We note that an equivalent way to define the amortization at state s is to compute the index  $g_s \geq 0$  that satisfies

$$
c(s)p(s) = \sum_{t \in T(s)} p(t) \cdot (g_s - w(t))^+
$$

<span id="page-39-0"></span>and then set  $b_{st} := (g_s - w(t))^+$  for each  $t \in T(s)$ . A pictorial representation of water filling amortization is shown in Figure [8.](#page-39-0)



Figure 8: For a state s with  $T(s) = \{t_1, t_2, t_2\}$ , the water filling algorithm will find the (unique)  $g_s \ge 0$  for which the highlighted area becomes  $c(s)$ . Then, it sets  $b_{st} = (g_s - w(t))^+$  and updates  $w(t) = \max(g_s, w(t))$ for all  $t \in T(s)$ .

Optimality of Water Filling Index Policy. Finally, we will combine everything to argue about the optimality of the water filling index policy, establishing Theorem [3.2.](#page-10-2)

**Theorem 3.2.** For any matroid-min-CICS instance  $\mathbb{I} = (\mathcal{M}_1, \cdots, \mathcal{M}_n, \mathcal{F})$  over Markov chains, the expected cost of the water filling index policy is equal to  $E[\min_{S \in \mathcal{F}} \sum_{i \in S} W^*_{\mathcal{M}_i}]$ . The policy is therefore optimal for instance I.

Proof. From Corollary [A.1,](#page-37-0) it suffices to argue that the water filling index policy, paired with the water filling amortization, satisfies both surrogate optimality and promise of payment.

We begin with surrogate optimality. By definition, the water filling index policy advances the chain of minimum water filling index; recall that the index of a state corresponds to the minimum surrogate cost among its terminal states. This ensures that while there is potential for some chain  $\mathcal{M}_i$  to realize the terminal state of minimum surrogate cost, the algorithm will keep advancing it. In other words, this algorithms ends up greedily accepting Markov chains with respect to the surrogate cost of their realized terminal states. Paired with the fact that a minimum cost basis of a matroid is always obtained by greedily adding the cheaper feasible element, this establishes surrogate optimality of the water filling index policy.

Next, we show that promise of payment also holds. Say that at any point the algorithm advances Markov chain  $\mathcal{M}_i$  with index  $I_i$  and reaches a state  $s_i \in S_i$  such that  $b_{s'_i t_i} > 0$  for some terminal state  $t_i \in T(s_i)$  and some ancestor  $s'_i$  of  $s_i$ . Since  $\mathcal{M}_i$  was advanced,  $I_i$  was the minimum index among all Markov chains; if we can prove that the index at state  $s_i$  is also  $I_i$ , then the algorithm will keep advancing  $\mathcal{M}_i$  and the promise of payment property will hold. Since  $b_{s_i't_i} > 0$  for some terminal state  $t_i \in T(s_i)$  and some ancestor  $s_i'$  of  $s_i$ , we know that when  $s'_i$  was being amortized,  $t_i$  was the terminal state in  $T(s'_i)$  with minimum current surrogate cost. By definition of the water filling amortization, this implies that  $t_i$  will continue to have the minimum surrogate cost among all the terminal states in  $T(s_i)$  for the rest of the amortization process, implying that the index of all states in the path from  $t_i$  to  $s'_i$  (including state  $s_i$ ) will be the same.  $\Box$ 

#### <span id="page-40-0"></span>A.2 Amortized Surrogate Costs for MDPs

In this section we formally prove our main result from Section [3,](#page-9-0) namely Theorem [3.4.](#page-11-4) We note that the proof mirrors our approach for proving the same result in the special case of Markov chains (Lemma [3.1\)](#page-9-2), and is enabled by the characterization of Lemma [3.3.](#page-11-1)

**Theorem 3.4.** For any matroid-min-CICS instance  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$ , the expected cost of the optimal adaptive policy is at least  $E\left[\min_{S \in \mathcal{F}} \sum_{i \in S} W_{\mathcal{M}_i}^*\right]$ .

*Proof.* Fix any algorithm (including the optimal adaptive policy) for instance  $\mathbb{I}$  and let ALG denote the (random) cost of this algorithm. Also, for all  $i \in [n]$ , let ALG(i) denote the (random) cost suffered by the algorithm from paying action costs and accepting terminal states in  $\mathcal{M}_i$ . Notice that  $\text{ALG} = \sum_{i=1}^n \text{ALG}(i)$ with probability 1. Finally, let  $X(i)$  be the random variable indicating whether the algorithm accepts a state from  $\mathcal{M}_i$  or not. To prove Theorem [3.4,](#page-11-4) we will argue that for all  $i \in [n]$ ,

<span id="page-40-1"></span>
$$
\mathcal{E}\left[\mathcal{A}\mathcal{L}\mathcal{G}(i)\right] \ge \mathcal{E}\left[X(i)\cdot W_{\mathcal{M}_i}^*\right].\tag{11}
$$

Notice that if this is true, then we immediately have that

$$
\mathrm{E}\left[\mathrm{ALG}\right] = \sum_{i=1}^{n} \mathrm{E}\left[\mathrm{ALG}(i)\right] \ge \sum_{i=1}^{n} \mathrm{E}\left[X(i) \cdot W_{\mathcal{M}_i}^*\right] = \mathrm{E}\left[\sum_{i=1}^{n} X(i) \cdot W_{\mathcal{M}_i}^*\right] \ge \mathrm{E}\left[\min_{S \in \mathcal{F}} \sum_{i \in S} W_{\mathcal{M}_i}^*\right]
$$

with the last inequality following from the fact that for any feasible algorithm, the set of accepted terminals  $S = \{i : X(i) = 1\}$  must be feasible (i.e.  $S \in \mathcal{F}$ ) with probability 1.

We now turn our attention to proving inequality  $(11)$ . We use A to denote the inner randomness of the algorithm and  $\mathcal{R}_i$  to denote the randomness of each MDP  $\mathcal{M}_i$  for all  $i \in [n]$ . Notice that conditioned on  $\mathcal{A}$ and  $\mathcal{R}_i$  for all  $i \in [n]$ , the outcome of the algorithm is deterministic. Now fix any  $i \in [n]$  and let

$$
\mathcal{R}^{-i} := \mathcal{A} \cup \big(\cup_{j \neq i} \mathcal{R}_j\big)
$$

encode all the randomness in the algorithm's run **except** from the realizations of  $\mathcal{M}_i$ . The key observation is that conditioned on  $\mathcal{R}^{-i}$ , all the actions that the algorithm takes in MDP  $\mathcal{M}_i$  are predetermined; in other words, the algorithm's trajectory on  $\mathcal{M}_i$  is fully described by some committing policy  $\pi_i = \pi_i(\mathcal{R}^{-i}) \in \mathcal{C}(\mathcal{M}_i)$ . Thus, we have that

<span id="page-40-2"></span>
$$
E\left[ALG(i)\right] = E_{\mathcal{R}^{-i}}\left[E_{\mathcal{R}_i}\left[ALG(i)|R^{-i}\right]\right] = E_{\mathcal{R}^{-i}}\left[\text{cost}(\pi_i)\right]
$$
\n(12)

where  $cost(\pi_i)$  is the expected cost of the algorithm, following the committing policy  $\pi_i = \pi_i(\mathcal{R}^{-i})$ , on MDP  $\mathcal{M}_i$ . The proof is the completed by the following generalization of Lemma [3.1,](#page-9-2) which is enabled from Lemma [3.3.](#page-11-1)

<span id="page-40-3"></span>**Claim 3.** Fix any MDP M and any committing policy  $\pi \in \mathcal{C}(\mathcal{M})$ . Then, the expected cost of any algorithm following  $\pi$  on M will be at least

$$
\mathrm{E}\left[X(\pi)\cdot W^*_{\mathcal{M}}\right]
$$

where  $X(\pi)$  is an indicator of whether the algorithm accepts a terminal state of M or not.

Notice that the above claim doesn't depend on the underlying CICS instance  $\mathbb{I}$ ; it simply states that conditioned on running a committing policy on some MDP and accepting a terminal state, the expected total cost spent on this MDP is lower bounded by the surrogate cost. Since this lower bound applies to all committing policies  $\pi \in \mathcal{C}(\mathcal{M})$ , coupled with equation [\(12\)](#page-40-2), it directly implies inequality [\(11\)](#page-40-1), as

$$
\mathrm{E}\left[\mathrm{ALG}(i)\right] = \mathrm{E}_{\mathcal{R}^{-i}}\left[\mathrm{cost}(\pi_i)\right] \geq \mathrm{E}_{R^{-i}}\left[\mathrm{E}\left[\left(X(i)|R^{-i}\right)\cdot W_{\mathcal{M}_i}^*\right]\right] = \mathrm{E}\left[X(i)\cdot W_{\mathcal{M}_i}^*\right],
$$

completing the proof.

**Proof of Claim [3.](#page-40-3)** Notice that since we are committing to running policy  $\pi \in \mathcal{C}(\mathcal{M})$  on M, we are essentially running some algorithm on the Markov chain  $\mathcal{M}^{\pi}$ . From Lemma [3.1](#page-9-2) and Theorem [3.2,](#page-10-2) we know that the contribution of Markov chain  $\mathcal{M}^{\pi}$  to the total cost will be at least

$$
\mathrm{E}\left[X(\pi) \cdot W_{\mathcal{M}^\pi}^*\right]
$$

and thus to prove the claim, we will need to show that

$$
\mathrm{E}\left[W_{\mathcal{M}^\pi}^*\right] \geq \mathrm{E}\left[W_{\mathcal{M}}^*\right]
$$

for all  $\pi \in \mathcal{C}(\mathcal{M})$ .

Let  $\mathcal{M} = (S, \sigma, A, c, \mathcal{D}, V, T)$  and recall that we use  $S_{\pi}$  and  $T_{\pi}$  to denote the state space and terminal states of  $\mathcal{M}^{\pi}$  and  $p_{\pi}$  to denote the implied distribution over  $T_{\pi}$ . Finally, we use  $c(s)$  to denote the cost of the unique action that  $\pi$  chooses at state  $s \in S_{\pi}$ . By definition of the water filling surrogate cost of a Markov chain, we have

$$
\mathcal{E}\left[W_{\mathcal{M}^{\pi}}^*\right] = \sum_{t \in T_{\pi}} p_{\pi}(t) \cdot W_{\mathcal{M}^{\pi}}^*(t) = \sum_{t \in T_{\pi}} p_{\pi}(t) \cdot \left(v(t) + \sum_{s \in S_{\pi}: t \in T_{\pi}(s)} b_{st}^*\right)
$$

where  $b_{st}^*$  are non-negative cost shares satisfying  $\sum_{t \in T_\pi(s)} p_\pi(t) b_{st}^* = p_\pi(s) c(s)$  for all  $s \in S_\pi$ .

From Lemma [3.3,](#page-11-1) there exists an amortized cost function  $\rho(\cdot)$  over the terminal states T of M and a set of non-negative cost shares  $\{b\}_{st}$  that satisfy action independence, cost sharing and cost dominance. Using these, we have

$$
E[W_{\mathcal{M}^{\pi}}^{*}] = \sum_{t \in T_{\pi}} p_{\pi}(t) \cdot \left(v(t) + \sum_{s \in S_{\pi}: t \in T_{\pi}(s)} b_{st}^{*}\right)
$$
  
\n
$$
= \sum_{t \in T_{\pi}} p_{\pi}(t) \cdot \left(E[\rho(t)] - \sum_{s \in S_{\pi}: t \in T_{\pi}(s)} b_{st} + \sum_{s \in S_{\pi}: t \in T_{\pi}(s)} b_{st}^{*}\right)
$$
 (Cost Sharing)  
\n
$$
= \sum_{t \in T_{\pi}} p_{\pi}(t) \cdot E[\rho(t)] + \sum_{s \in S_{\pi}} \sum_{t \in T_{\pi}(s)} p_{\pi}(t) \cdot (b_{st}^{*} - b_{st})
$$
  
\n
$$
= \sum_{t \in T_{\pi}} p_{\pi}(t) \cdot E[\rho(t)] + \sum_{s \in S_{\pi}} \left(p_{\pi}(s)c(s) - \sum_{t \in T_{\pi}(s)} p_{\pi}(t) \cdot b_{st}\right)
$$
  
\n
$$
\geq \sum_{t \in T_{\pi}} p_{\pi}(t) \cdot E[\rho(t)]
$$
 (Cost Dominance)  
\n
$$
= E[W_{\mathcal{M}}^{*}].
$$
 (Action Independence)



#### <span id="page-41-0"></span>A.3 Second Order Stochastic Dominance

Finally, in this section we provide a proof for Lemma [3.5.](#page-13-0) Once again, we note that this result is standard; here, we provide our own constructive proof for the sake of completeness and building intuition on how the water filling surrogate cost of an MDP is obtained. For simplicity, we refer to Lemma [3.5](#page-13-0) as the Stochastic Dominance Lemma, henceforth SDL.

**Lemma 3.5.** (Second Order Stochastic Dominance.) Let  $X, Z$  be discrete random variables that satisfy the property  $E[\min\{y, X\}] \le E[\min\{y, Z\}]$  for all  $y \in \mathbb{R}$ . There exists a mapping  $m : \text{supp}(Z) \mapsto \Delta(\text{supp}(X))$ from the support of  $Z$  to distributions over the support of  $X$  such that:

1. X is obtained by sampling from  $m(z)$  for a randomly sampled  $z \sim Z$ .

#### 2. For all  $z \in \text{support}(Z)$ , it holds that  $E[m(z)] \leq z$ .

*Proof.* We will say that  $X \preceq Z$  if there exists a mapping  $m : supp(Z) \mapsto \Delta(supp(X))$  such that the two conditions of the SDL hold. We associate each random variable W with a function  $f_W(y) := \mathbb{E}[\min\{y, W\}]$ over  $y \in \mathbb{R}$ . In other words, we want to prove that

$$
f_X(y) \le f_Z(y) \quad \forall y \in \mathbb{R} \Longrightarrow X \preceq Z.
$$

We first prove **transitivity** of our condition. In other words, if  $X, Y, Z$  are discrete random variables such that  $X \preceq Y$  and  $Y \preceq Z$ , we also have that  $X \preceq Z$ . The proof is immediate; let  $m_1 : \text{supp}(Z) \mapsto \Delta(\text{supp}(Y))$ be the corresponding mapping for  $Y \preceq Z$  and  $m_2 : supp(Y) \mapsto \Delta(\text{supp}(X))$  be the corresponding mapping for  $X \preceq Y$ . Then, the composition mapping  $m := m_2 \circ m_1$  that maps each  $z \in \text{supp}(Z)$  to a random realizations of  $m_2(y)$  for a randomly sampled  $y \sim m_1(z)$  immediately satisfies both conditions and yields  $X \preceq Z$ . Formally, for each  $x \in \text{supp}(X)$  we have

$$
\Pr[X = x] = \sum_{y,z} \Pr[Z = z] \cdot \Pr[m_1(z) = y] \cdot \Pr[m_2(y) = x] = \sum_{z} \Pr[Z = z] \cdot \Pr[m(z) = x]
$$

and for each  $z \in \text{supp}(Z)$  we have

$$
E[m(z)] = \sum_{y} Pr[m_1(z) = y] \cdot E[m_2(y)] \le \sum_{y} Pr[m_1(z) = y] \cdot y = E[m_1(z)] \le z
$$

and thus transitivity of the  $\preceq$  operator is established.

We will now proceed to the **main proof**. Fix the random variables X and Z such that  $f_X(y) \leq f_Z(y)$ for all  $y \in \mathbb{R}$  and order their support sets so that

$$
\mathcal{X} := \text{support}(X) = \{x_1, \cdots, x_M\}
$$

and

$$
\mathcal{Z} := \text{support}(Z) = \{z_1, \cdots, z_N\}
$$

with  $x_1 < x_2 < \cdots < x_M$  and  $z_1 < z_2 < \cdots < z_N$ . We also use  $p_i^X := \Pr[X = x_i]$  for all  $i \in [N]$  and  $p_i^Z := \Pr[Z = Z_i]$  for all  $i \in [M]$  to denote the corresponding probabilities, with

$$
\sum_{i=1}^{M} p_i^X = \sum_{i=1}^{N} p_i^Z = 1.
$$

We will say that X and Z **agree up to index i**, if  $x_j = z_j$  and  $p_j^X = p_j^Z$  for all  $j < i$ . Conventionally, we say that any two random variables will agree up to index 1 according to this definition. We will structure our proof of the SDL as an induction on the maximum index that  $X$  and  $Z$  agree up to. In particular, we break-down our proof in the following two steps:

- 1. (Induction Base). If X and Z agree up to index  $M$ , the SDL holds.
- 2. (Induction Step). If X and Z agree up to index  $i \in [M-1]$ , there exists a random variable Z' such that
	- (a)  $Z'$  and X agree up to index  $(i + 1)$ .
	- (b)  $Z' \preceq Z$ .
	- (c) For all  $y \in \mathbb{R}$ ,  $f_X(y) \leq f_{Z'}(y)$ .

Before proving each of these two claims, let's see how they naturally construct an inductive proof for the SDL. Initially, we have that by definition,  $X$  and  $Z$  agree up to index 1. We can then apply our second claim (i.e. the induction step) to obtain a random variable  $Z_1 \preceq Z$  that agrees with X up to index 2 and satisfies  $f_X(y) \le f_{Z_1}(y)$  for all  $y \in \mathbb{R}$ . We can then re-apply the induction step to obtain a random variable  $Z_2 \preceq Z_1$ that agrees with X up to index 3 and satisfies  $f_X(y) \le f_{Z_2}(y)$  for all  $y \in \mathbb{R}$ . We keep applying the induction step for as long as we can, until we obtain a random variable  $Z_{M-1} \preceq Z_{M-2} \preceq \cdots \preceq Z_1 \preceq Z$  that agrees with X up to index M and satisfies  $f_X(y) \le f_{Z_{M-1}}(y)$  for all  $y \in \mathbb{R}$ . We then proceed to use our first claim (i.e. the induction base) to show that  $X \preceq Z_{M-1}$ . Finally, we use the transitivity of operator  $\preceq$  in order to obtain  $X \preceq Z$ , which concludes the proof.

**Proof of Induction Base.** Assume that X and Z agree up to index M; this means that  $x_i = z_i$  and  $p_i^X = p_i^Z$  for all  $i < M$ . Then, consider the deterministic mapping  $m(z_i) = x_i$  if  $i < M$  and  $m(z_i) = x_M$ if  $i \geq M$ . Since  $x_M$  is the last point in the support X, sampling  $z \sim Z$  and outputting  $m(z)$  is clearly equivalent to sampling  $x \sim X$ . For the expectation condition, we need to show that  $m(z_i) = x_M \leq z_i$  for all  $i \geq M$ . For  $y = x_M$ , we have that  $f_X(x_M) \leq f_Z(x_M)$ . By definition:

$$
f_X(x_M) = \sum_{i=1}^{M} p_i^X \cdot x_i = \sum_{i=1}^{M-1} p_i^X \cdot x_i + p_M^X \cdot x_M
$$

and

$$
f_Z(x_M) = \sum_{i=1}^{M-1} p_i^Z \cdot z_i + \sum_{i=M}^{N} p_i^Z \cdot \min(z_i, x_M).
$$

Observe that the sums for  $i \in [M-1]$  are equal by the agreement assumption and also  $\sum_{i=M}^{N} p_i^Z = p_M^X$ . Thus, to satisfy  $f_X(x_M) \leq f_Z(x_M)$  we would need  $x_M \leq \min(x_M, z_i)$  for all  $i \geq M$  or equivalently that  $x_M \leq z_M < z_{M+1} < \cdots < z_N$  and the proof follows.

**Proof of Induction Step.** We will now prove the induction step. Assume that the random variables X and Z agree up to index i for some  $i \in [M-1]$ ; if they also agree up to index  $(i + 1)$  the step follows for  $Z' = Z$  since clearly  $Z \preceq Z$ . If they don't agree up to index  $(i + 1)$ , then by  $f_X(y) \le f_Z(y)$  for all  $y \in \mathbb{R}$ it must necessarily be the case that either  $x_i < z_i$  or  $(x_i = z_i \text{ and } p_i^X > p_i^Z)$ . In any case, we deduce that  $f_X(y) = f_Z(y)$  for all  $y \leq x_i$  and  $\lim_{y \to x_i^+} f_X(y) < \lim_{y \to x_i^+} f_Z(y)$ .

Now, let  $\ell(y)$  denote the unique straight line that passes through points  $(x_i, f_X(x_i))$  and  $(x_{i+1}, f_X(x_{i+1}))$ , that is:

$$
\ell(y) = \frac{f_X(x_{i+1}) - f_X(x_i)}{x_{i+1} - x_i} \cdot (y - x_i) + f_X(x_i).
$$

<span id="page-43-0"></span>Since  $f_Z(y)$  is clearly concave,  $\ell(y)$  will intersect it in at most two points; one of them is the point  $(x_i, f_Z(x_i)) = (x_i, f_X(x_i))$  and the other point will necessarily be  $(s, f_Z(s))$  for some  $s \geq x_{i+1}$ . A pictorial representation is given in Figure [9.](#page-43-0)



Figure 9: The induction step. Random variables X and Z agree up to index 2. The line  $\ell(y)$  extends the line-segment of  $f_X(y)$  from  $x_2$  to  $x_3$  and cuts  $f_Z(y)$  on some  $y = s \ge x_3$ . Let  $h(y)$  denote the highlighted curve. The random variable Z' that has curve  $f_{Z'}(y) = h(y)$  agrees with X up to index 3 and satisfies  $f_{Z'}(y) \ge f_X(y)$  for all  $y \in \mathbb{R}$ . Here,  $J = \{2,3\}$  denotes the set of indices of  $f_Z$ 's break-points that lie in  $(x_2, s)$ .

Now, consider the curve  $h(y) = \min\{\ell(y), f_Z(y)\}\.$  By construction, this curve satisfies  $f_X(y) \leq h(y)$  for all  $y \in \mathbb{R}$ . Furthermore, if it is the case that there exists some random variable Z' such that  $f'_Z(y) = h(y)$ , then it would be the case that Z' and X agree up to index  $(i + 1)$ . Thus, to complete the proof, we need to show that such a random variable Z' not only exists, but also satisfies  $Z' \preceq Z$ . This will require it's own type of induction, so we will state it an prove it as a separate claim (Claim [4\)](#page-44-0) to ease the presentation. Notice that once this claim is proven, the proof of the SDL is completed.  $\Box$ 

<span id="page-44-0"></span>**Claim 4.** For any discrete random variable Z with curve  $f_Z(y) = E[\min\{y, Z\}]$  and any line  $\ell(y)$  intersecting  $f_Z(y)$  at exactly two points  $a < b$ , there always exists discrete random variable Z' such that  $Z' \preceq Z$  and  $f_{Z'}(y) = \min\{f_Z(y), \ell(y)\}.$ 

*Proof.* Once again, we denote  $\mathcal{Z} := \text{support}(Z) = \{z_1, \dots, z_N\}$  and assume  $z_1 < z_2 < \dots < z_N$ . We also use  $p_i^Z := \Pr[Z = z_i]$  for  $i \in [N]$ . Notice that the function  $f_Z(y)$  is piece-wise linear, with breakpoints precisely at  $z_i \in \mathcal{Z}$ . Furthermore, the slope at the interval  $(-\infty, z_i]$  is 1, the slope at any interval  $[z_i, z_{i+1}]$  for  $i \in [N-1]$ is  $1 - \sum_{j \leq i} p_j^Z$  and the slope at the interval  $[z_N, \infty)$  is 0. Finally, for each point  $z_i \in \mathcal{Z}$ , the difference of the slope of  $f_Z(y)$  on the segment to its left minus the slope of  $f_Z(y)$  on the segment to its right equals the probability  $p_i^Z$ .

We will begin by designing a useful **gadget**. This gadget  $G(Z, a, b)$  takes as input a discrete random variable Z and two parameters  $a < b$  such that there exists index  $i \in [N]$  with  $z_{i-1} \leq a < z_i < b \leq z_{i+1}$ (we denote  $z_0 = -\infty$  and  $z_{N+1} = +\infty$ ) and returns a random variable Z' that is obtained by mapping each point  $z_i \in \mathcal{Z}$  with  $j \neq i$  to itself, and mapping point  $z_i$  to point a with some probability  $\lambda$  and to point b with probability  $1 - \lambda$ . Clearly, support $(Z') = \{a, b\} \cup \mathcal{Z} \setminus \{z_i\}$ . Let  $\ell_{ab}(y)$  be the line passing through points  $(a, f_Z(a))$  and  $(b, f_Z(b))$  and let s be the slope of this line. Furthermore, let  $s_1$  be the slope of  $f_Z(y)$ at the interval  $[a, z_i]$  and  $s_2$  be its slope at the interval  $[z_i, b]$ ; by definition,  $s_1 - s_2 = p_i^Z$ . Also, note that  $s_1 > s > s_2$  by concavity. Then, by using mapping probability

$$
\lambda:=\frac{s_1-s}{s_1-s_2}
$$

it is not hard to see that  $f_{Z'}(y) = \min\{f_Z(y), \ell_{ab}(y)\}\$ , since we maintain the probability mass at all points  $z_j \neq z_i$  and furthermore we have  $\Pr[Z' = a] = p_i^Z \cdot \lambda = s_1 - s$  and  $\Pr[Z' = b] = p_i^Z \cdot (1 - \lambda) = s - s_2$ . Furthermore, it is also not hard to see that  $Z' \preceq Z$ ; we only need to verify that

$$
\lambda \cdot a + (1 - \lambda) \cdot b \le z_i
$$

or equivalently (by substituting  $\lambda$ 's definition) that  $s(b-a) \leq s_1(z_i-a) + s_2(b-z_i)$ ; this always holds with equality due to the definition of  $s_1, s_2$  and s.

We are now ready to prove the claim. Let  $J = \{z_i \in (a, b)\}\;$ ; this is the set of points  $z_i \in \mathcal{Z}$  for which  $f_Z(z_i) > \ell(z_i)$ . Notice that the gadget  $G(Z, a, b)$  already proves the claim for the special case of  $|J| = 1$ . We will now inductively apply the gadget to prove the general case. Let  $z_i$  be the minimum point in J; we begin by applying the gadget  $G(z_i, a, z_{i+1})$  to obtain a new random variable Z'; this is allowed since  $z_i$  is the unique point of Z in the interval  $(a, z_{i+1})$ . By our construction of the gadget and concavity of  $f_Z(y)$ , we have that  $Z' \preceq Z$  and also that  $f_{Z'}(y) \ge \min\{f_Z(y), \ell(y)\}\)$  for all  $y \in \mathbb{R}$ . Furthermore, the corresponding J-interval for Z' will now have one less point; thus, we can inductively keep applying our gadget and by transitivity of the  $\preceq$  operator the claim follows. A pictorial proof is shown in Figure [10](#page-45-0)

 $\Box$ 



<span id="page-45-0"></span>Figure 10: Here,  $J = \{z_2, z_3\}$ . At each step, we isolate the leftmost point of  $f_Z(y)$  that is dominated by  $\ell(y)$ and use our gadget to distribute it between  $a$  and the next point. Eventually, we recover a random variable with optimality curve  $\min\{f_Z(y), \ell(y)\}.$ 

# <span id="page-46-0"></span>B Omitted Proofs from Section [4](#page-15-0)

In this section, we present the omitted proof of Theorem [4.1](#page-16-1) from Section [4.](#page-15-0) We restate all our results for the reader's convenience.

**Theorem 4.1.** Let  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  be any instance of matroid-min-CICS, where each MDP  $\mathcal{M}_i$ admits an  $\alpha$ -local approximation under some committing policy  $\pi_i \in \mathcal{C}(\mathcal{M}_i)$ . Then,

$$
\mathbf{E}\left[\min_{S\in\mathcal{F}}\sum_{i\in S}W_{\mathcal{M}_i^{\pi_i}}^*\right]\leq \alpha\cdot\mathbf{E}\left[\min_{S\in\mathcal{F}}\sum_{i\in S}W_{\mathcal{M}_i}^*\right].
$$

*Proof.* Let  $z := [W^*_{\mathcal{M}^\pi_2}, \cdots W^*_{\mathcal{M}^\pi_n}]$  encode the surrogate costs of all Markov chains  $\mathcal{M}_i^{\pi_i}$  with the exception of  $\mathcal{M}_1^{\pi_1}$ . Then, we have

$$
\begin{aligned} \mathbf{E}\left[\underset{S\in\mathcal{F}}{\min}\sum_{i\in S}W_{\mathcal{M}_i^{\pi_i}}^*\right] &=\mathbf{E}_z\left[\mathbf{E}_{W_{\mathcal{M}_1^{\pi_1}}} \left[\underset{S\in\mathcal{F}}{\min}\sum_{i\in S}W_{\mathcal{M}_i^{\pi_i}}^*\right]\right] \\ &=\mathbf{E}_z\left[\mathbf{E}_{W_{\mathcal{M}_1^{\pi_1}}} \left[\underset{S\in\mathcal{F}}{\min}\left(W_{\mathcal{M}_1^{\pi_1}}^*\cdot \mathbbm{1}[1\in S]+\sum_{i\in S\backslash\{1\}}W_{\mathcal{M}_i^{\pi_i}}^*\right)\right]\right] \\ &=\mathbf{E}_z\left[\mathbf{E}_{W_{\mathcal{M}_1^{\pi_1}}} \left[\underset{S\in\mathcal{F}:1\notin S}{\min}\sum_{i\in S}W_{\mathcal{M}_i^{\pi_i}}^*\ ,\ W_{\mathcal{M}_1^{\pi_1}}^*+\underset{S:1\notin S;S\cup\{1\}\in\mathcal{F}}{\min}\sum_{i\in S}W_{\mathcal{M}_i^{\pi_i}}^*\right)\right]\end{aligned}
$$

Now, let  $f_1(z) := \min_{S \in \mathcal{F}: 1 \notin S} \sum_{i \in S} W^*_{\mathcal{M}_i^{\pi_i}}$  and  $f_2(z) := \min_{S: 1 \notin S; S \cup \{1\} \in \mathcal{F}} \sum_{i \in S} W^*_{\mathcal{M}_i^{\pi_i}}$ . Note that both quantities depend only on z and not on  $W^*_{\mathcal{M}_1^{\pi_1}}$ . Then, we have

$$
\begin{split}\n\mathbf{E}\left[\min_{S\in\mathcal{F}}\sum_{i\in S}W_{\mathcal{M}_{i}^{\pi_{i}}}^{*}\right] &= \mathbf{E}_{z}\left[\mathbf{E}_{W_{\mathcal{M}_{1}^{\pi_{1}}^{\pi_{1}}}}\left[\min\left(f_{1}(z) \ , \ W_{\mathcal{M}_{1}^{\pi_{1}}}^{*}+f_{2}(z)\right)\right]\right] \\
&= \mathbf{E}_{z}\left[f_{2}(z) + \mathbf{E}_{W_{\mathcal{M}_{1}^{\pi_{1}}^{\pi_{1}}}}\left[\min\left(f_{1}(z) - f_{2}(z) \ , \ W_{\mathcal{M}_{1}^{\pi_{1}}}^{*}\right)\right]\right] \\
&\leq \mathbf{E}_{z}\left[f_{2}(z) + \mathbf{E}_{W_{\mathcal{M}_{1}}^{\pi_{1}}}\left[\min\left(f_{1}(z) - f_{2}(z) \ , \ \alpha \cdot W_{\mathcal{M}_{1}}^{*}\right)\right]\right] \quad \text{(Local Approximation)} \\
&= \mathbf{E}_{z}\left[\mathbf{E}_{W_{\mathcal{M}_{1}}^{\pi_{1}}}\left[\min\left(f_{1}(z) \ , \ \alpha \cdot W_{\mathcal{M}_{1}}^{*}+f_{2}(z)\right)\right]\right] \\
&= \mathbf{E}\left[\min_{S\in\mathcal{F}}\sum_{i\in S}\tilde{W}_{i}\right]\n\end{split}
$$

where  $\tilde{W}_1 = \alpha \cdot W^*_{\mathcal{M}_1}$  and  $\tilde{W}_i := W^*_{\mathcal{M}_i^{\pi_i}}$  for all  $i \neq 1$ . Thus, we have substituted  $W^*_{\mathcal{M}_1^{\pi_1}}$  with  $\alpha \cdot W^*_{\mathcal{M}_1}$ . The proof is completed by repeating the same process for all other  $i \neq 1$ .  $\Box$ 

## <span id="page-47-0"></span>C Omitted Proofs from Section [5](#page-17-0)

In this chapter of the appendix, we present all omitted proofs from Section [5,](#page-17-0) in particular the proof of Lemma [5.2.](#page-18-1) We restate all our results for the reader's convenience.

**Lemma 5.2.** Consider any instance  $\mathbb{I} = (\mathcal{B}_1, \cdots, \mathcal{B}_n, \mathcal{F})$  of matroid-PBPI and partition the n boxes into two sets p

$$
O:=\left\{i\in[n]:\frac{c_i^o}{c_i^p}\leq 1+\frac{c_i^p}{c_i^o}\right\}
$$

and  $P = [n] \setminus O$ . The policy that commits to directly opening the boxes in O and peeking before opening the boxes in P achieves a  $\phi$ -approximation to the optimal adaptive policy.

Proof. The proof relies on the crucial fact that opening and peeking into a PI box reveals precisely the same information to the decision maker; in particular, the value of the box. We construct a policy that commits to directly opening boxes  $i \in O$  and peeking before opening boxes  $i \in P$ , while simultaneously mimicking the optimal adaptive policy. In particular, fix any box  $i$ .

- If the optimal never interacts with box  $i$ , neither does our policy.
- If the optimal directly opens box i, then (i) if  $i \in O$  our policy also opens it and never peeks into it, and (ii) if  $i \in P$  our policy first peeks into the box and then immediately opens it.
- If the optimal first peeks into box i, then (i) if  $i \in P$  our policy also peeks into it (and then opens it whenever the optimal decides to open it), and (ii) if  $i \in O$  our policy directly opens it and never peeks into it.
- If the optimal selects box  $i$ , so does our policy.

Observe that at any point the optimal and our policy have the exact same information, so we can keep mimicking the optimal decision tree. Furthermore, our policy clearly respects the given commitment. Finally, whenever the optimal selects a box, we can do the same as the set of our policy's opened boxes is always a superset of the optimal's opened boxes. This ensures feasibility of our algorithm under any combinatorial constraint F.

Thus, the only difference between the costs of our policy and the optimal is due to differences in the selected actions. In particular, if  $i \in P$  then if the optimal peeks into or ignores the box then so does the algorithm, with the worst case being the optimal directly opening the box. In that case, the optimal pays  $c_i^o$ whereas our policy pays  $c_i^p + c_i^o$ . On the other hand, if  $i \in O$ , then the worst case is if the optimal peeks into the box and decides not to open it; in that case the optimal pays  $c_i^p$  whereas our algorithm pays  $c_i^o$ . Finally, our algorithm pays precisely the same cost as the optimal for accepting boxes.

Combining everything, we conclude that our policy achieves an

$$
\alpha := \max\left(\, \max_{i \in O}(\frac{c_i^o}{c_i^p}) \ ,\ \max_{i \in P}(\frac{c_i^p + c_i^o}{c_i^o})\right)
$$

approximation to the optimal adaptive policy. For each box  $i \in [n]$ , let  $\lambda_i = c_i^p/c_i^o \in (0,1)$  and observe that by definition of the partition sets we have that  $1/\lambda_i \leq 1 + \lambda_i$  if and only if  $i \in O$ . Thus, we obtain that

$$
\alpha \leq \max_i \min(1 + \lambda_i, \frac{1}{\lambda_i}) \leq \max_{x \in (0,1)} \min(1 + x, \frac{1}{x}) = \phi.
$$



### <span id="page-48-0"></span>D Omitted Proofs from Section [6](#page-21-0)

In this chapter of the appendix, we present all omitted proofs from Section [6,](#page-21-0) in particular the proof of Theorem [6.2.](#page-21-3) We restate all our results for the reader's convenience.

**Theorem 6.2.** For any  $\alpha \geq 1$ , there exists a WS alternative M that does not admit an  $\alpha$ -pointwise approximation.

For ease of notation, let  $k := \alpha + 1$  and  $B := 2^{k^2}$ . We consider the alternative with weighing cost  $c = 1$ and random cost X that is continuously distributed in interval  $[1, B]$  and has two point masses on 0 and  $(k+1)B$ , namely:

$$
X = \begin{cases} 0 & \text{with probability } 1 - \frac{1}{k} \\ x \in [1, B] & \text{with density } f(x) = \frac{1}{kx^2} \\ (k+1)B & \text{with probability } \frac{1}{k} \end{cases}
$$

Note that since  $B > k > 1$  and  $\int_{x=1}^{B} \frac{1}{kx^2} dx = \frac{1}{k} - \frac{1}{kB}$ , X is indeed a valid random variable. We proceed by computing the relevant parameters of X.

- The expected value of X is  $\mu = \mathbb{E}[X] = k + 1 + \frac{1}{k}$  by definition. Thus,  $\mu \in [k+1, k+2]$ .
- The g-index of M satisfies  $g \in [1, \frac{k}{k-1}]$ . Define  $\operatorname{exc}(z) := \operatorname{E}[(z-X)^+]$ . The claim follows by noting that  $\text{exc}(\cdot)$  is non-decreasing;  $\text{exc}(1) = 1 - \frac{1}{k} < 1$  and  $\text{exc}(\frac{k}{k-1}) > 1$ ; whereas  $\text{exc}(g) = 1$  by definition.
- The *h*-index of *M* is  $h = B$ , since  $E[(X B)^+] = \frac{1}{k} \cdot [(k+1)B B] = 1$ .

By definition of  $\rho^*(x)$ , we have that  $\rho^*(0) = g$ ,  $\rho^*(k+1)B = B$  and  $\rho^*(x) = \max(g, x)$  for all  $x \in [1, B]$ . We will now show that no committing policy can achieve an  $\alpha'$ -pointwise approximation for the alternative  $(X, 1)$  for any  $\alpha' < \alpha$ .

Fix any committing policy  $\pi$  and let  $\ell(x) := E[X \in I]$  where  $I \in \mathcal{I}^{\pi}$  is the unique interval of the policy's partition that contains x. Clearly,  $\ell(x)$  is a lower bound to  $\rho^{\pi}(x)$  and it is also a non-decreasing function. Thus, in order to prove impossibility of pointwise approximation for any  $\alpha' < \alpha$ , it suffices to show that there exists some  $x \in \mathcal{X}$  for which

$$
\frac{\ell(x)}{\rho^*(x)} \ge \alpha.
$$

We begin by considering the committing policy  $\pi$  that does not perform any weighings. Then, we simply have  $\ell(x) = \rho^{\pi}(x) = \mu$  for all  $x \in \mathcal{X}$  and since  $\rho^*(x) \geq g$  for all  $x \in X$ , and the claim follows from  $\alpha \leq \mu/g$ . Next, we consider any committing policy that performs weighings, and let  $t$  be the maximum threshold that it uses. Clearly,  $t < (k+1)B$  otherwise there is no point in the weighing. This means that the final interval in  $\mathcal{I}^{\pi}$  will be  $I_{\infty} := (t, \infty)$ . Let  $\mu_{\infty} := \mu(I_{\infty}) = \mathbb{E}[X | X > t]$ . We distinguish between the following cases:

- 1. If  $t \geq B$ , then  $\mu_{\infty} = (k+1)B$  and  $\rho^*(t) \leq h = B$ . In that case, the ratio is at least  $k > \alpha$ .
- 2. If  $t \le 1$ , then  $\mu_{\infty} \ge E[X|X \ge 1] = k^2 + k + 1$  and  $\rho^*(t) \le \rho^*(1) = \min(h, \max(g, 1)) = g$ . In that case, the ratio is at least  $k^2 - k > \alpha$ .
- 3. Finally, if  $t \in (1, B)$ , then  $\Pr[X \ge t] = \frac{1}{kt}$  and thus  $\mu_{\infty} \ge kt$ . Since  $\rho^*(t) = \max(g, t)$ , the ratio is at least  $k - 1 = \alpha$ .

Thus, in any case there exists some x for which  $\ell(x) \geq \rho^*(x)$ . As already mentioned, by the fact that both mappings are non-decreasing and by definition of pointwise-approximation, this proves Theorem [6.2.](#page-21-3)

### <span id="page-49-0"></span>E The Maximization Setting

In this section, we describe how our entire framework extends to the maximization setting under matroid feasibility constraints. Our objective will be to restate our claims from Sections [3](#page-9-0) and [4,](#page-15-0) as they were only established with respect to the matroid-min-CICS problem. Since the proofs follow exactly the same steps, rather than re-proving all our results, we simply discuss the differences, where there are any.

#### <span id="page-49-1"></span>E.1 Amortization for Markov Chains

The cost amortization of a Markov chain is defined in the same manner, with the difference that instead of increasing the terminal state values to obtain surrogate costs, we now decrease them to obtain surrogate values. Furthermore, the index of a state now corresponds to the maximum surrogate value among its downwards terminal states instead of the minimum. Formally:

Definition 12 (Cost amortization for maximization settings). A cost amortization for a Markov chain  $\mathcal{M} = (S, \sigma, A, c, \mathcal{D}, V, T)$  is a non-negative vector  $b = \{b_{st}\}_{s \in S, t \in T(s)}$  with the property that

$$
\sum_{t \in T(s)} p(t)b_{st} = p(s)c(s)
$$

for all states  $s \in S$ . Based on an amortization, we define:

- The amortized value of a terminal state  $t \in T$  as  $\rho_b(t) := v(t) \sum_{s:t \in T(s)} b_{st}$ .
- The surrogate value of the Markov chain M as the random variable  $\rho_{\mathcal{M},b}$  that takes on value  $\rho_b(t)$  for  $t \in T$  with probability  $p(t)$ .
- The index of a state  $s \in S$  of the Markov chain M as  $I_{\mathcal{M},b}(s) = \max_{t \in T(s)} \rho_b(t)$ .

Intuitively, we postpone the payment of the action costs until a terminal state is accepted, in which case a smaller (compared to the original value) surrogate value is collected. From this, we follow precisely the same steps as in the proof of Lemma [3.1](#page-9-2) to upper bound the the utility of any algorithm for matroid-max-CICS through the surrogate values.

**Lemma E.1** (Markov chain upper upper for maximization settings). Consider any matroid-max-CICS instance  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  over Markov chains and let b<sub>i</sub> be any cost amortization of  $\mathcal{M}_i$  with corresponding surrogate cost  $\rho_i := \rho_{\mathcal{M}_i,b_i}$  for all  $i \in [n]$ . Then, the expected utility of any algorithm for  $\mathbb{I}$  is at  $most \mathop{\to} [\max_{S \in \mathcal{F}} \sum_{i \in S} \rho_i].$ 

Up next, we extend our definition of water filling amortization. Like before, we start from the terminal states, and define their total cost share to be equal to their value  $v(t)$  and then proceed over non-terminal states in decreasing order of level. The difference is that each non-terminal state  $s \in S$  distributes its total cost  $c(s)$  across its downstream terminals  $T(s)$ , starting from the terminals with the **maximum** current total value, until the equation  $\sum_{t \in T(s)} p(t) b_{st} = p(s)c(s)$  is satisfied. In other words, instead of increasing the cost of the minimum-cost terminal, we now decrease the value of the maximum-value terminal. Thus, a more suitable term that we will be using in the maximization setting will be water draining. We now use  $W^*_{\mathcal{M}}$  to denote the water draining surrogate value of a Markov chain  $\mathcal{M}$ , and  $I^*_{\mathcal{M}}(s)$  to denote the water draining index of a state  $s$  in  $\mathcal{M}$ .

Through this amortization, we once again define the corresponding index based policy; naturally, the policy will now select to advance the feasible Markov chain of maximum index.

Definition 13 (Water Draining Index policy). The Water Draining Index policy for a matroid-max-CICS instance  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  chooses at every step the Markov chain  $i^* = \operatorname{argmax}_{i \in \mathcal{F}_S} I_i^*(s_i)$ , where  $s_i$ is the current state of each Markov chain  $\mathcal{M}_i$ ; S is the set of terminated (selected) Markov chains; and  $\mathcal{F}_S = \{i : S \cup \{i\} \in \mathcal{F}\}.$ 

From the same observations as in the proof of Lemma [3.1,](#page-9-2) it immediately follows that this algorithm achieves two desired properties: it always selects the feasible set of Markov chains with maximum total surrogate value, and whenever it advances a state  $s \in S$  that contributes to the surrogate value of a terminal state  $t \in T(s)$  (i.e.  $b_{st} > 0$ ) and nature realizes t, the algorithm will select it. From this, and the fact that the algorithm that greedily adds the maximum weight feasible element is optimal for the matroid independent set problem, the following counterpart to Theorem [3.2](#page-10-2) follows, establishing the optimality of the Water Draining Index policy for matroid-max-CICS.

<span id="page-50-2"></span>**Theorem E.2.** For any matroid-max-CICS instance  $\mathbb{I} = (\mathcal{M}_1, \cdots, \mathcal{M}_n, \mathcal{F})$  over Markov chains, the expected utility of the water draining index policy is equal to  $E[\max_{S \in \mathcal{F}} \sum_{i \in S} W^*_{\mathcal{M}_i}]$ . The policy is therefore optimal for instance I.

#### <span id="page-50-0"></span>E.2 Optimality Curves

The definition of a local game  $(M, y)$  naturally extends to the maximization setting; at any step, the decision maker can either advance the Markov chain  $M$  at a cost (until it reaches a terminal state in which case it may collect its value and terminate), or collect the reward  $y$  of the outside option and terminate. Like before, we use  $f_{\mathcal{M}}(y)$  to denote the utility of the optimal adaptive policy in this game. From Theorem [E.2,](#page-50-2) we immediately have that

$$
f_{\mathcal{M}}(y) = \mathrm{E}\left[\max\{y, W_{\mathcal{M}}^*\}\right].
$$

From this expression, we obtain that the CDF of the water draining surrogate value  $W^*_{\mathcal{M}}$  can be derived from the optimality curve as  $\frac{d}{dy} f_{\mathcal{M}}(y)$ . This in turn allows us to define water draining surrogate values for arbitrary MDPs.

**Definition 14** (Water draining surrogate values for MDPs). Let  $M$  be an MDP with optimality curve  $f_M$ . The Water Draining Surrogate Value for M is the random variable  $W^*_{\mathcal{M}}$  generated by picking a value from the CDF  $\frac{d}{dy} f_{\mathcal{M}}(y)$ . That is,  $W^*_{\mathcal{M}}$  is the random variable satisfying  $f_{\mathcal{M}}(y) = E[\max(y, W^*_{\mathcal{M}})]$  for all  $y \in \mathbb{R}$ .

#### <span id="page-50-1"></span>E.3 Amortization for MDPs

We will now use the definition of the water draining surrogate values to prove the following counterpart of Lemma [3.3](#page-11-1) that characterizes the water draining surrogate value of an MDP. Since this is the most technical proof of the framework and there are a few arguments that change in the maximization setting, we provide a proof sketch for the result.

<span id="page-50-3"></span>**Lemma E.3.** For any MDP  $M = (S, \sigma, A, c, D, V, T)$ , there exists an amortized cost function  $\rho : T \mapsto \Delta(\mathbb{R})$ mapping terminal states to distributions over costs and a non-negative cost sharing vector  $b = \{b_{st}\}_{s \in S, t \in T(s)}$ , such that for all committing policies  $\pi \in \mathcal{C}(\mathcal{M})$ , generating a Markov chain  $\mathcal{M}^{\pi}$  with states  $S_{\pi} \subseteq S$ , terminal states  $T_{\pi} \subseteq T \cap S_{\pi}$ , and a distribution  $p_{\pi}$  over them, the following properties hold:

- 1. Action Independence. The water draining surrogate cost  $W^*_{\mathcal{M}}$  corresponds to sampling a terminal state  $t \sim p_{\pi}$  and then sampling from distribution  $\rho(t)$ .
- 2. Cost Sharing. For all  $t \in T_{\pi}$ ,  $E[\rho(t)] = v(t) \sum_{s \in S_{\pi}: t \in T_{\pi}(s)} b_{st}$ .
- 3. Cost Dominance. For all  $s \in S_{\pi}$ ,  $\sum_{t \in T_{\pi}(s)} p_{\pi}(t) b_{st} \leq p_{\pi}(s) c(a)$  for the unique action  $a \in A(s)$ chosen by  $\pi$ .

Proof. Notice that the only change with respect to the statement of Lemma [3.3](#page-11-1) for the minimization setting is the negative sign in the cost sharing property. Our proof will mirror the proof of Lemma [3.3,](#page-11-1) and we use the same notation throughout. Once again, the proof proceeds by induction on the horizon of the MDP; the  $H = 1$  case remains trivial.

For the amortization of the action costs  $c(a_i)$ , we will now define  $g_i$  to be the solution to equation

$$
\mathbb{E}\left[Z_j\right] - c(a_j) = \mathbb{E}\left[\min\{g_j, Z_j\}\right]
$$

and by defining  $\hat{Z}_j := \min\{g_j, Z_j\}$ , we have

$$
\max\{y, \mathbf{E}\left[\max\{y, Z_j\}\right] - c(a_j)\} = \mathbf{E}\left[\max\{y, \hat{Z}_j\}\right]
$$

which in turn allows us to argue that

$$
\mathrm{E}\left[\max\{y,W^*_\mathcal{M}\}\right] \geq \mathrm{E}\left[\max\{y,\hat{Z}_j\}\right]
$$

for all  $y \in \mathbb{R}$  and  $j \in [k]$ .

Using the identity  $\max(a, b) = -\min(-a, -b)$ , this implies that for all  $j \in [k]$ , the random variable  $(-\hat{Z}_j)$  second-order stochastically dominates the random variable  $(-W^*_{\mathcal{M}})$ . From Lemma [3.5,](#page-13-0) this allows us to obtain mappings  $m_j : \text{supp}(\hat{Z}_j) \mapsto \Delta(\text{supp}(W^*_{{\mathcal M}}))$  such that:

1. For each  $j \in [k]$ ,  $W^*_{\mathcal{M}}$  can be obtain by sampling a  $z \sim \hat{Z}_j$  and then sampling from  $m_j(z)$ .

2. For each  $z \in \text{supp}(\hat{Z}_j)$ , we have  $E[m_j(z)] \geq z$ .

We can now define the amortized cost function  $\rho$  and the cost sharing vector b. Fix some  $t \in T$ , and let  $j \in [k]$  and  $s \in S$  be the indices of the unique action  $a_j$  and state  $s \in R_j$  such that  $t \in T(s)$ . We define

$$
\rho(t) := m_j(\min\{g_j, \rho_s(t)\})
$$

and

$$
b_{\sigma t} := \mathcal{E}\left[\rho_s(t)\right] - \mathcal{E}\left[\rho(t)\right]
$$

and for all other  $s' \in S \setminus \{\sigma\}$  with  $t \in T(s')$ , we use the same cost share  $b_{s't} = b_{s't}^s$  that was used in  $\mathcal{M}_s$ . The proof of the three properties follows precisely the same steps as Lemma [3.3.](#page-11-1)

From Lemma [E.3,](#page-50-3) we immediately obtain the following counterpart of Theorem [3.4;](#page-11-4) the proof follows exactly the same steps as the proof of Theorem [3.4](#page-11-4) that we presented in Appendix [A,](#page-36-0) and is thus omitted.

<span id="page-51-1"></span>**Theorem E.4.** For any matroid-max-CICS instance  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$ , the expected utility of the optimal adaptive policy is at most  $E\left[\max_{S \in \mathcal{F}} \sum_{i \in S} W_{\mathcal{M}_i}^*\right]$ .

#### <span id="page-51-0"></span>E.4 Local Approximation

Finally, we note that our notion of local approximation seamlessly extends to the maximization setting by simply changing the inequality order. In particular:

**Definition 15** (Local approximation for maximization settings). Let  $M$  be any MDP. We say that a committing policy  $\pi \in \mathcal{C}(\mathcal{M})$  α-locally approximates M for some  $\alpha \in (0,1]$  if

$$
f_{\mathcal{M}^{\pi}}(\alpha y) \geq \alpha \cdot f_{\mathcal{M}}(y) \quad \forall y \in \mathbb{R}.
$$

Notice that in the maximization setting, we have an  $\alpha$ -local approximations for  $\alpha \in (0,1]$ . By following exactly the same steps as in the proof of Theorem [4.1,](#page-16-1) the following composition theorem is immediate:

<span id="page-51-2"></span>**Theorem E.5** (Composition theorem for maximization settings). Let  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F})$  be any instance of matroid-max-CICS, where each MDP  $\mathcal{M}_i$  admits an  $\alpha$ -local approximation under some committing policy  $\pi_i \in \mathcal{C}(\mathcal{M}_i)$ . Then,

$$
\mathbf{E}\left[\max_{S\in\mathcal{F}}\sum_{i\in S}W_{\mathcal{M}_i^{\pi_i}}^*\right] \geq \alpha \cdot \mathbf{E}\left[\max_{S\in\mathcal{F}}\sum_{i\in S}W_{\mathcal{M}_i}^*\right].
$$

By combining Theorem [E.5](#page-51-2) with the optimality of the Water Draining Index policy (Theorem [E.2\)](#page-50-2) and the upper bound of Theorem [E.4,](#page-51-1) we obtain a way to efficiently approximate the optimal solution for matroidmax-CICS assuming that the underlying MDPs achieve good local approximation guarantees.

### <span id="page-52-0"></span>F The Combinatorial Setting

In this section, we describe how our entire framework extends to combinatorial settings beyond the case of matroids. We will state our results in full generality and distinguish between minimization and maximization whenever needed. The framework we consider is based on [\[Singla, 2017\]](#page-35-0) and its followup [\[Gupta et al., 2019\]](#page-35-2), where the authors develop a technique for combinatorial selection over Markov chains. In this section, we show that under local approximation, their results can be seamlessly extended to arbitrary MDPs.

**Definition 16** (CICS). A Costly Information with Combinatorial Inspection (CICS) instance  $\mathbb{I}$  is defined with respect to a set of Costly Information MDPs  $\{\mathcal{M}_i\}_{i=1}^n$ , a feasibility constraint  $\mathcal{F} \subseteq 2^{[n]}$  and a function  $h : \mathcal{F} \to \mathbb{R}$ . At each step, an algorithm chooses one of the MDPs and advances it through one of its actions. The game terminates once the algorithm accepts a feasible<sup>[11](#page-52-2)</sup> set  $S \in \mathcal{F}$  of MDPs.

For a specific run of the algorithm, let A denote the set of all actions the algorithm took,  $S \in \mathcal{F}$  denote the set of terminated MDPs and  $T$  be the corresponding set of terminals it accepted. Then, the total cost of the algorithm for this run in the minimization setting is given by

$$
\sum_{t \in T} v(t) + h(S) + \sum_{a \in A} c(a)
$$

and the total utility of the algorithm for the run in the maximization setting is given by

$$
\sum_{t \in T} v(t) + h(S) - \sum_{a \in A} c(a).
$$

The optimal adaptive policy in the minimization (resp. maximization) setting is the policy of minimum (resp. maximum) expected cost (resp. utility).

We note that there are two differences with respect to the matroid setting. The first is that  $\mathcal F$  can now be any arbitrary set of constraints. In fact, we won't even have to assume that the set is downwards or upwards close, as long as the algorithms always ensure that they accept a feasible set of MDPs. The second is the addition of the function  $h(\cdot)$ ; this is a function that encodes an extra cost (or reward, depending on the setting) that does not depend on the precise terminals of the MDPs that were selected, but only on the set of terminated MDPs.

The notions of amortization, water filling/draining, optimality curves, surrogate costs and local approximation apply to each MDP separately, and thus are independent of the underlying combinatorial setting. Therefore, all these definitions extend to the combinatorial setting without change. Our first contribution is to extend Theorem [3.4](#page-11-4) beyond the matroid setting and prove that the performance of the optimal adaptive policy in any combinatorial setting is bounded by the surrogate costs of the underlying MDPs. We note that while this result was known in the single-selection setting, we are the first to prove it for the general combinatorial setting.

<span id="page-52-3"></span>**Theorem F.1.** Let  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F}, h)$  be any instance of CICS. For each  $i \in [n]$ , let  $W^*_{\mathcal{M}_i}$  be the water filling (resp. water draining) surrogate costs in the minimization (resp. maximization) setting. Let OPT be the expected cost (resp. utility) of the optimal adaptive policy. Then:

- 1. For the minimization setting,  $\text{OPT} \geq \text{E} \left[ \min_{S \in \mathcal{F}} (\sum_{i \in S} W_{\mathcal{M}_i}^* + h(S)) \right]$ .
- 2. For the maximization setting,  $\text{OPT} \leq \text{E} \left[ \max_{S \in \mathcal{F}} (\sum_{i \in S} W_{\mathcal{M}_i}^* + h(S)) \right]$ .

Our second contribution is to show that local approximation continues to imply composition results even in the combinatorial setting. In particular, we prove the following extension of Theorem [4.1.](#page-16-1)

<span id="page-52-1"></span>**Theorem F.2.** Let  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F}, h)$  be any instance of CICS. For each  $i \in [n]$ , let  $W^*_{\mathcal{M}_i}$  be the water filling (resp. water draining) surrogate costs in the minimization (resp. maximization) setting. Finally, for each  $i \in [n]$ , let  $\pi_i \in \mathcal{C}(\mathcal{M}_i)$  be some committing policy that  $\alpha$ -locally approximates  $\mathcal{M}_i$ . Then:

<span id="page-52-2"></span> $11$ We assume that the algorithm will always ensure that this happens, i.e. when having already accepted a set of MDPs  $S$ , it will never accept another MDP *i* for which  $S \cup \{i\} \cup A \notin \mathcal{F}$  for all  $A \subseteq [n]$ .

1. For the minimization setting,  $E\left[\min_{S \in \mathcal{F}} (\sum_{i \in S} W_{\mathcal{M}_i^{\pi_i}}^* + h(S))\right] \leq \alpha \cdot E\left[\min_{S \in \mathcal{F}} (\sum_{i \in S} W_{\mathcal{M}_i}^* + h(S))\right]$ . 2. For the maximization setting,  $E\left[\max_{S\in\mathcal{F}}(\sum_{i\in S}W^*_{\mathcal{M}_i^{\pi_i}}+h(S))\right]\geq \alpha \cdot E\left[\max_{S\in\mathcal{F}}(\sum_{i\in S}W^*_{\mathcal{M}_i}+h(S))\right]$ .

By combining Theorem [F.1](#page-52-3) with Theorem [F.2,](#page-52-1) and assuming that our MDPs admit local approximation guarantees, we are left with the task of optimizing over the CICS instance that is generated by the committing policies; observe that this is now an instance over Markov chains. In the case of matroid constraints, we showed that we can always efficiently achieve this via the water filling/draining index policy. However, depending on the combinatorial constraint, efficient optimization might not be possible – consider for example the case where each MDP is a single terminal state corresponding to a set of elements and we need to select a minimum cost set cover.

The final key to the puzzle will be a way of efficiently approximating the optimal policy for a CICS instance  $\mathbb{I} = (\mathcal{M}_1, \dots, \mathcal{M}_n, \mathcal{F}, h)$  over Markov chains. This is precisely the setting that is considered by [Gupta et al.](#page-35-2) [\[2019\]](#page-35-2). In this work, it is shown that a sufficient condition to get an efficient  $\beta$ -approximation to the optimal policy for a CICS instance over Markov chains is for the underlying pair  $(\mathcal{F}, h)$  to admit a β-approximate frugal algorithm. We defer the reader to [Gupta et al.](#page-35-2) [\[2019\]](#page-35-2) for the full details. Here, we will just mention some examples of such combinatorial settings:

- Matching constraints in the maximization setting admit a 2-approximate frugal algorithm.
- Facility location constraints in the minimization setting admit a 1.861-approximate frugal algorithm.
- Set cover constraints in the minimization setting admit a  $\min(f, \log n)$ -approximate frugal algorithm where  $f$  is the maximum number of sets in which a ground element is present.

Combining everything, we obtain the following result for the combinatorial setting:

**Corollary F.3.** Let  $\mathbb{I} = (\mathcal{M}_1, \cdots, \mathcal{M}_n, \mathcal{F}, h)$  be any instance of CICS such that:

- 1. Each MDP  $\mathcal{M}_i$  admits an  $\alpha$ -local approximation.
- 2. The combinatorial setting  $(F, h)$  admits a  $\beta$ -frugal algorithm.

Then, we can efficiently obtain an  $(\alpha \cdot \beta)$ -approximation to the optimal adaptive policy for  $\mathbb{I}$ .

We are left with the task of proving our extended Theorems [F.1](#page-52-3) and [F.2.](#page-52-1) It is not hard to see that both proofs are identical to their matroid counterparts and are thus omitted. In particular:

- 1. The proof of Theorem [F.1](#page-52-3) follows exactly the same steps as the proofs of Theorem [3.4](#page-11-4) (for the minimization setting) and Theorem [E.4](#page-51-1) (for the maximization setting) that were presented in Appendix [A](#page-36-0) and Appendix [E](#page-49-0) respectively. In these proofs, we separately bounded the cost/utility contribution of each MDP by the corresponding surrogate costs/values and simply invoked the feasibility of the optimal adaptive policy at the end. Thus, the exact same proofs imply Theorem [F.1](#page-52-3) for any feasibility constraint F and any cost/reward function  $h(\cdot)$ .
- 2. The proof of Theorem [F.2](#page-52-1) follows exactly the same steps as the proofs of Theorem [4.1](#page-16-1) (for the minimization setting) and Theorem [E.5](#page-51-2) that were presented in Appendix [B](#page-46-0) and Appendix [E](#page-49-0) respectively. In particular, none of these proofs used the fact that  $\mathcal F$  is a matroid constraint at any point, and it is also straightforward to see that they immediately extend for any cost/reward function  $h(\cdot)$ .