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STEIN'S METHOD AND GENERAL CLOCKS: DIFFUSION APPROXIMATION OF THE G/G/1 WORKLOAD

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We begin developing the theory of the generator comparison approach of Stein's method for continuous-time Markov processes where jumps are driven by clocks having general distributions, as opposed to exponential distributions. This paper handles models with a single general clock.

Using the workload process in the $G/G/1$ queueing system as a driving example, we develop two variants of the generator comparison approach for models with a single general clock: the original, which we call the *limiting* approach, and the recently proposed *prelimit* approach. The approaches are duals of one another, yielding distinct bounds on the diffusion approximation error of the steady-state workload.

We also contribute to the theory of heavy-traffic approximations for the $G/G/1$ system. Under some assumptions on the interarrival time distribution, the prelimit approach allows us to bound the diffusion approximation error in terms of $G/G/1$ model primitives. For example, when the interarrival time has a nonincreasing hazard rate that is bounded from above, we show that the diffusion approximation error of the expected workload is bounded in terms of the first three moments of the interarrival and service-time distributions, as well as the upper bound on the interarrival hazard rate.

1. Introduction. The generator comparison approach of Stein's method is a powerful technique for comparing stationary distributions of Markov processes, and has been widely applied to continuous-time Markov chains (CTMCs) and discrete-time Markov processes. Due to a gap in the theory, it has yet to be applied to continuous-time Markov processes where jumps are driven by clocks having general distributions. We say that these models have general clocks, in contrast to CTMCs where clocks are exponentially distributed. We begin to fill this gap by developing the generator comparison approach for systems with a single general clock.

We illustrate the approach using the single-server queue with general interarrival and service-time distributions, known as the $G/G/1$ system. We focus on the workload process, which consists of the remaining workload and the residual interarrival time. The former decays at a unit rate and increases when new customers arrive to the system, and the latter is the general clock tracking the time until the next arrival. We bound the error of the exponential approximation of the workload, and our approach generalizes to other models with a single general clock.

A byproduct of our analysis is a novel upper bound on the expected busy period duration that requires only the first two moments of the interarrival and service-time distributions to be finite. The only other bound that we are aware of is due to [32] and requires three finite moments.

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We work with two variants of the generator comparison approach: the original, attributed to [5, 3] and [23], and the prelimit approach, recently recently proposed by [9]. The original approach starts from the Poisson equation of the approximating diffusion, also called the Stein equation, while the prelimit approach uses the Poisson equation of the Markov process to be approximated—the prelimit. We refer to the former as the *limiting* approach for ease of reference.

One of our goals with this work was to compare the fruits of the prelimit and limiting approaches, to highlight each of their relative strengths. The limiting approach yields a bound on the exponential approximation error that involves the expected equilibrium idle period length, which can be restated in terms of the first two moments of the descending ladder heights of the random walk corresponding to the customer waiting times. Except for special cases, there is no simple expression for this quantity in terms of $G/G/1$ model primitives; see [31, 40] for efforts to analyze the idle period. The same term also appears in the error bound of Theorem 1.2 in [22], where the authors analyze the $G/G/1$ waiting time approximation using Stein's method for the exponential distribution.

The prelimit approach yields an entirely different expression for the exponential approximation error that depends on the density at zero of the idle period duration; see the second-order Stein factor bounds of Lemma 12. When the interarrival distribution has a density and its hazard rate is bounded from above, and either (a) the hazard rate is nondecreasing or (b) the hazard rate is bounded from below, we bound the approximation error using only the moments of the interarrival and service-time distributions and the bounds on the hazard rate. As a sample of our main results, let U and S denote the interarrival and service-time distributions, respectively, let $\lambda = 1/\mathbb{E}U$, let $\rho < 1$ be the system utilization, and let V be the steady-state workload. Provided that U has a bounded and nonincreasing hazard rate, we show in Theorem 2 that

$$\left|(1-\rho)\mathbb{E}V - \lambda\mathbb{E}(U-S)^2/2\right| \leq (1-\rho)\overline{C}_3,$$

where $\overline{C}_3$ is known explicitly and depends on the interarrival hazard rate at zero and the first three moments of U and S.

Cases (a) and (b) represent a nontrivial class of interarrival distributions. Generalizing our error bounds to broader classes of hazard rates is possible at the cost of increased complexity in bounding the third-order Stein factors, as we would need to adapt the result of Lemma 14 to the new hazard rate. Handling interarrival distributions with point masses is more challenging, because the density at zero of the idle period cannot be written in terms of the interarrival distribution hazard rate; see Remark 4.

The prelimit and limiting approaches are duals of one another. Summarizing their differences, the limiting approach starts from the Poisson equation for the exponential distribution, which has well-known Stein factor bounds. However, the limiting approach considers the expectation of the generator difference with respect to the distribution of the workload, resulting in the complicated idle period term. In contrast, the prelimit approach starts from the Poisson equation for the workload, which has complicated Stein factor bounds, but considers the expectation of the generator difference with respect to the simple exponential distribution. It is notable that combining the results of both the prelimit and limiting approaches yields stronger bounds than either approach does on its own; see the discussion following Theorem 2.

Lastly, we comment on the added technical challenge of applying both approaches to a model with a general clock. The limiting approach requires using the Palm inversion formula to extract the generator of the approximating diffusion from the jump component of the workload process stationary equation. The prelimit approach does not require Palm calculus. The key is to average the Poisson equation, which depends on both the workload and residual interarrival time, over the stationary distribution of the residual time. This averaged Poisson equation depends on both the time average of its solution, the value function, as well as its expectation at jump times. Exploiting the structure of the value function, we relate both of these terms without using the Palm inversion formula. It is interesting to note that the value function naturally lies in the domain of the workload generator; i.e., the jump terms in the stationary equation for that function equal zero. This is in contrast to the test functions used in [34, 35, 10], which had to be carefully engineered in order to have the jump terms equal to zero.

1.1. *Literature review.* The generator comparison approach of Stein's method is a powerful technique for comparing Markov process stationary distributions. Though Stein's method dates back to the seminal paper of [38], the connection between Stein's method and Markov procesess is attributed to [5, 3] and [23]. A particularly rich application domain for the generator comparison approach is the field of queueing theory. The approach was applied to birth-death processes by [14]. Later, more complex queueing systems were analyzed by [25, 39, 24], though it is notable that these authors used the main elements of the generator approach without being aware of the rich literature on Stein's method. A few years later, the generator comparison approach was popularized in queueing theory by [13, 12] in the setting of diffusion approximations, and by [41, 42, 21] in the setting of mean-field models. Since then, Stein's method and, specifically, the generator comparison approach has been an active area of research in the queueing community.

The seminal work of [29, 30] initiated a wave of research into upper and lower bounds on the expected waiting time in the $G/G/1$ system; for surveys of the numerous existing bounds, see [20, 40]. Since the expected waiting time has a well-known relationship to the expected workload [1, Corollary X.3.5], bounds on one quantity translate to bounds on the other.

Though existing existing waiting time bounds are tight as the system utilization tends to one, most of them do not quantify the gap between the bound and expected waiting time. Some exceptions include [8, Equation (7)], which uses the transform method to give an expression for the moments of the waiting time. In particular, they provide an approximation the expected waiting time and a corresponding upper bound on the error that is of $O(1/\log(1-\rho))$, though the constant in their error bound is not explicitly known. In comparison, we provide an approximation with an $O(1)$ error bound (does not increase as $\rho \to 1$) and an explicit constant. Another approximation for the expected waiting time that is $o(1)$ accurate (error goes to zero as $\rho \to 1$) is stated following [6, Theorem 3.3], though we cannot find a proof. Lastly, another related research direction is on extremal queues by [15, 16, 17, 18], where authors identify interarrival and service-time distributions for which the bounds on expected waiting time are tight.

In addition to the numerous bounds on the expected waiting time, there have been several applications of Stein's method to the single-server queue. Bounds on the exponential approximation of the customer waiting time in the $G/G/1$ system were obtained using equilibrium couplings by [22], where the authors follow the approaches of [36, 37] and exploit the fact that the waiting time is a convolution of a geometrically distributed number of i.i.d. random variables. They also get error bounds for the $M/G/1$ system using the generator comparison approach. Their $G/G/1$ bound depends on the expected equilibrium idle period, which is exponentially distributed in the case of the $M/G/1$ model since arrivals follow a Poisson process.

In [28], the authors apply the generator comparison approach to the workload process of the $M/G/1+GI$ system—the single-server queue with general patience-time distribution and Poisson arrivals. They focus on establishing diffusion approximation error bounds that are universal across various patience-time distributions and system loads.

The queue-length of a discrete-time $G/G/1$ system is considered by [43], where the authors bound the error of the exponential approximation. Analogous to the equilibrium idle period in a continuous-time model, they need to bound the second moment of the steady-state unused service. They do this by assuming that the number of service completions in a single time slot is bounded.

Lastly, a simple application of the prelimit approach to the $M/M/1$ queue length can be found in [9]. In addition to steady-state approximations, [7] establishes process-level rates of convergence to the diffusion approximation for the $M/M/1$ and $M/M/\infty$ systems. More recently, [4] develops an approach for approximating random processes with Gaussian processes, and applies it to the $G/G/\infty$ system.

2. The $G/G/1$ workload process. Consider a single-server queueing system operating under a first-in-first-out (FIFO) service discipline. Let U and S be random variables having the interarrival and service time distributions, respectively. Let $G(x) = \mathbb{P}(U \leq x)$, and let

$$\lambda = 1/\mathbb{E}U \quad \text{and} \quad \rho = \lambda \mathbb{E}S,$$

be the arrival rate and system utilization, respectively. We assume that the first three moments of U and S are finite, though in some places we need the fourth moment of S to be finite as well.

Let $V(t)$ be the workload in the system and $R(t)$ be the remaining time until the next arrival at time $t \geq 0$; we call $R(t)$ the residual time. Similarly, let $A(t)$ be the elapsed time since the last arrival prior to $t \geq 0$ or simply the age at time t. Set

$$\delta = (1-\rho) \quad \text{and} \quad X(t) = \delta V(t),$$

and consider the right continuous with left limits (RCLL) workload process

$$\{Z(t) = (X(t), R(t)) : t \geq 0\},$$

which is a piecewise-deterministic Markov process with state space

$$\mathbb{S} = \{(x,r) \in \mathbb{R}^2_+ : x \geq 0, r > 0\}.$$

Note that $(x, 0) \notin \mathbb{S}$ for $x \geq 0$ because the workload process is RCLL.

Let $\bar{B}$ denote the duration of a busy period initialized by an arrival to an empty system, and let $\bar{I}$ denote the duration of the subsequent idle period. We let B_0 be the length of the initial busy period, with the convention that $B_0 = 0$ if $X(0) = 0$. We also let B_n, $n \geq 1$, be the lengths of the subsequent busy periods, which are i.i.d. and have the same distribution as $\bar{B}$. We also let I_n, $n \geq 0$, be the duration of the idle period following B_n, and note that I_n, $n \geq 1$ are i.i.d. with the same distribution as $\bar{I}$. When $\rho < 1$, [1, Propositions X.1.3 and X.3.1] say that

$$\mathbb{E}\bar{I} = \frac{1-\rho}{\rho}\mathbb{E}\bar{B} < \infty. \tag{2.1}$$

Furthermore, note that

$$\mathbb{E}\bar{B} = \lim_{\epsilon \downarrow 0} \mathbb{E}_{0,\epsilon} B_0 = \mathbb{E}(\mathbb{E}_{\delta S, U} B_0),$$

where $\mathbb{E}_{(x,r)}(\cdot) = \mathbb{E}(\cdot | Z(0) = (x,r))$ and the outer expectation on the right-hand side is with respect to the distributions of U and S.

The workload process is a regenerative process, with regeneration happening at those instances when a customer arrives to an empty system. Going forward we assume that

$$\rho < 1 \text{ and } U \text{ is nonlattice}. \tag{2.2}$$

When a customer arrives to an empty system at $t = 0$ and (2.2) holds, [1, Corollary X.3.3] guarantees the existence of a limiting steady-state distribution for $\{Z(t) : t \geq 0\}$. We let (X, R) have this distribution and note that

$$\mathbb{P}(R \leq x) = \lambda \int_0^x (1 - G(t))dt, \quad x \geq 0, \tag{2.3}$$

since R is the steady-state residual interarrival time.

We now argue that the expected time to regeneration is finite under (2.2) given any initial condition $(x, r) \in \mathbb{S}$. Namely, we show that

$$\mathbb{E}_{x,r} B_0 \leq \mathbb{E}\big(\mathbb{E}_{x+\delta S, U} B_0\big) < \infty. \tag{2.4}$$

The first inequality is true because the busy period is made longer if an arrival happens immediately. For the second inequality, let us treat the (unscaled) initial work x/δ as low-priority work and all other work as high-priority work. The low-priority work is cleared when there is no high priority work; i.e., during the idle periods in a system with only high-priority work. Thus, the initial low-priority workload is cleared when the cumulative time spent without high-priority work exceeds x/δ. If N_x is the number of idle periods required for this, then $\mathbb{E}N_x < \infty$ by [26], and by Wald's identity, the expected end of the N_xth idle period equals $\mathbb{E}N_x \mathbb{E}(\bar{B} + \bar{I})$, which is finite, implying (2.4).

We conclude this section by introducing some notation. At times, we will need to consider expected values with respect to some variables but not others. For example, given random variables T_n, $n \geq 1$, and some subset of them $T_{n_1}, \ldots, T_{n_k}$, we write

$$\mathbb{E}^{T_{n_1}, \ldots, T_{n_k}}(f(T_1, \ldots, T_n))$$

to denote an expectation with respect $T_{n_1}, \ldots, T_{n_k}$ only. We still write $\mathbb{E}(\cdot)$ to denote the expectation over all random quantities inside the parentheses.

We let Lip(1) denote the set of Lipschitz-1 functions $f : \mathbb{R} \to \mathbb{R}$, and for any integer $k \geq 1$, we define

$$\mathcal{M}_k = \Big\{ f : \mathbb{R} \to \mathbb{R} : \Big\| \frac{\partial^j f}{\partial x^j} \Big\| \leq 1, \ 1 \leq j \leq k \Big\}, \tag{2.5}$$

where $\|\cdot\|$ is the infinity norm.

3. The limiting generator comparison approach.

Let Y be exponentially distributed with rate $2\theta/\sigma^2$ and, for all $f \in C^2(\mathbb{R})$, define

$$G_Y f(x) = -\theta f'(x) + \frac{1}{2}\sigma^2 f''(x), \quad x \in \mathbb{R}. \tag{3.1}$$

Then Y is the stationary distribution of a one-dimensional reflected Brownian motion with generator G_Y [27]. This connection is meant to add context, and is not used anywhere in this paper. Recalling that $\delta = (1 - \rho)$, the following is the main result of this section.

THEOREM 1. *Let Y be exponentially distributed with rate $2/(\lambda \mathbb{E}(S - \rho U)^2)$. Then*

$$\sup_{h \in \text{Lip}(1)} \big|\mathbb{E}h(X) - \mathbb{E}h(Y)\big| \leq (1 - \rho)\Big((2 + \rho)\frac{\mathbb{E}U^2}{2\mathbb{E}U} + \rho \frac{\mathbb{E}\bar{I}^2}{2\mathbb{E}\bar{I}} + 2\mathbb{E}S + 4\frac{\mathbb{E}\,|S - \rho U|^3}{\mathbb{E}(S - \rho U)^2}\Big).$$

We prove Theorem 1 at the end of this section after introducing all then necessary ingredients. Given $h : \mathbb{R} \to \mathbb{R}$, consider the differential equation

$$G_Y f_h(x) = \mathbb{E}h(Y) - h(x), \quad x \in \mathbb{R}. \tag{3.2}$$

The following lemma bounds $f_h(x)$ and its derivatives, and is proved in Section 3.2.

LEMMA 1. *The solution to* (3.2) *satisfies* $f_h'(0)=0$. *Furthermore, if* $h\in \mathrm{Lip}(1)$, *then* $f_h'''(x)$ *is absolutely continuous and*

$$\|f_h''\|\le 1/\theta \quad \textit{and} \quad \|f_h'''\|\le 4/\sigma^2,$$

and, as a consequence,

$$\left|f_h'(x)\right|\le |x|/\theta \quad \textit{and} \quad |f_h(x)|\le \frac{1}{2}x^2/\theta.$$

Setting $x=X$ in (3.2) and taking expectations yields

$$\mathbb{E}h(Y)-\mathbb{E}h(X)=-\theta\mathbb{E}f_h'(X)+\frac{1}{2}\sigma^2\mathbb{E}f_h''(X). \tag{3.3}$$

All expectations on the right-hand side are well defined by Lemma 1 and the fact that $\mathbb{E}X<\infty$ [1, Theorems X.2.1 and X.3.4].

We compare the right-hand side of (3.3) to the stationary equation for the workload, also called the basic adjoint relationship (BAR) [10], which says that if $Z(0)$ is initialized according to the stationary distribution (X,R), then for all sufficiently regular functions $f:\mathbb{S}\to\mathbb{R}$,

$$0=\mathbb{E}\bigg(\int_0^t \big(-\delta 1(X(s)>0)\partial_x f(Z(s))-\partial_r f(Z(s))\big)ds\bigg)+\mathbb{E}\bigg(\sum_{m=1}^{\infty}1(\tau_m\le t)\Delta f(Z(\tau_m))\bigg),$$

where τ_m is the time of the mth arrival and $\Delta f(Z(t))=f(Z(t))-f(Z(t-))$. The following lemma, proved in Section 3.1, contains the BAR for $f(Z(t))=f_h(X(t)-\delta\rho R(t))$.

LEMMA 2. *For any* $h\in\mathrm{Lip}(1)$,

$$\begin{aligned}0=&-\delta\mathbb{E}\big((1(X>0)-\rho)f_h'(X-\delta\rho R)\big)+\frac{1}{2}\delta^2\mathbb{E}(S-\rho U)^2\sum_{m=1}^{\infty}\mathbb{E}\big(1(\tau_m\le 1)f_h''(X(\tau_m-))\big)\\ &+\sum_{m=1}^{\infty}\mathbb{E}\Big(1(\tau_m\le 1)\int_0^{\delta(S-\rho U)}(\delta(S-\rho U)-v)\int_0^v f_h'''(X(\tau_m-)+u)du dv\Big).\end{aligned} \tag{3.4}$$

REMARK 1. We work with the BAR for $f_h(X(t)-\delta\rho R(t))$ instead of $f_h(X(t))$ because it simplifies the algebra. Using $f_h(X(t))$ is possible but creates a term involving $f_h'(X(\tau_m-))$ in (3.4), which then necessitates more applications of the Palm inversion formula (Lemma 3).

The BAR (3.4) suggests what the values of θ and σ^2 in (3.3) should be. For example, the first term in the BAR satisfies

$$-\delta\mathbb{E}\big((1(X>0)-\rho)f_h'(X-\delta\rho R)\big)\approx-\delta\mathbb{E}\big((1-\rho)f_h'(X)\big)=-\delta^2\mathbb{E}f_h'(X),$$

so we choose $\theta=\delta^2$. The choice of σ^2 comes from the term involving $f_h''(X(\tau_m-))$ but is not yet apparent, because we need to use the Palm inversion formula to relate this term to $\mathbb{E}f_h''(X)$.

The following is a special case of the Palm inversion formula [2, Equation (1.2.25)]. We provide an elementary proof in Section 3.3 that does not use Palm calculus. Let $U_m=\tau_{m+1}-\tau_m$ be the interarrival time of the customer arriving at τ_{m+1} and let S_m be the workload brought by the customer arriving at τ_m.

LEMMA 3. *For any $h \in \text{Lip}(1)$,*

$$\mathbb{E} f_h''(X) = \sum_{m=1}^{\infty} \mathbb{E}\Big(1(\tau_m \le 1) \int_0^{U_m} f_h''\big((X(\tau_m-) + \delta S_m - \delta u)^+\big) du\Big)$$

PROOF OF THEOREM 1. Fix $h \in \text{Lip}(1)$. With $\theta = \delta^2$ and $\sigma^2 = \delta^2 \lambda \mathbb{E}(S - \rho U)^2$, subtract (3.4) from (3.3) to get

$$\begin{aligned} &\mathbb{E}h(Y) - \mathbb{E}h(X) \\ =& -\delta^2\big(\mathbb{E}f_h'(X) - \mathbb{E}f_h'(X - \delta\rho R)\big) - \delta\mathbb{E}\big(1(X=0) f_h'(-\delta\rho R)\big) \\ &+ \frac{1}{2}\delta^2 \mathbb{E}(S-\rho U)^2\Big(\lambda \mathbb{E} f_h''(X) - \sum_{m=1}^{\infty} \mathbb{E}\big(1(\tau_m \le 1) f_h''(X(\tau_m-))\big)\Big) \\ &+ \sum_{m=1}^{\infty} \mathbb{E}\Big(1(\tau_m \le 1) \int_0^{\delta(S-\rho U)} (\delta(S-\rho U) - v) \int_0^v f_h'''(X(\tau_m-) + u) du dv\Big). \end{aligned} \tag{3.5}$$

The result follows once we show that

$$\delta^2 \left|\mathbb{E}f_h'(X) - \mathbb{E}f_h'(X - \delta\rho R)\right| \le \delta\rho \frac{\mathbb{E}U^2}{2\mathbb{E}U}, \tag{3.6}$$

$$\delta \mathbb{E}\left|1(X=0) f_h'(-\delta\rho R)\right| \le \delta\rho \frac{\mathbb{E}\bar{I}^2}{2\mathbb{E}\bar{I}}, \tag{3.7}$$

$$\frac{1}{2}\delta^2 \mathbb{E}(S-\rho U)^2 \Big|\lambda \mathbb{E} f_h''(X) - \sum_{m=1}^{\infty} \mathbb{E}\big(1(\tau_m \le 1) f_h''(X(\tau_m-))\big)\Big| \le 2\delta\Big(\mathbb{E}S + \frac{\mathbb{E}U^2}{2\mathbb{E}U}\Big), \tag{3.8}$$

$$\sum_{m=1}^{\infty} \mathbb{E}\Big|1(\tau_m \le 1) \int_0^{\delta(S-\rho U)} (\delta(S-\rho U) - v) \int_0^v f_h'''(X(\tau_m-) + u) du dv\Big| \le 4\delta \frac{\mathbb{E}|S-\rho U|^3}{\mathbb{E}(S-\rho U)^2}. \tag{3.9}$$

We begin with (3.6). Observe that

$$\delta^2 \left|\mathbb{E}f_h'(X) - \mathbb{E}f_h'(X - \delta\rho R)\right| \le \delta^3 \rho \mathbb{E}R \|f_h''\| \le \delta^3 \rho \frac{1}{2} \lambda \mathbb{E}U^2 \frac{1}{\delta^2} = \delta\rho \frac{\mathbb{E}U^2}{2\mathbb{E}U},$$

where the last inequality is due to Lemma 1. To prove (3.7), note that $f_h'(0) = 0$ by Lemma 1, and therefore

$$\delta\mathbb{E}\left|1(X=0) f_h'(-\delta\rho R)\right| \le \delta^2 \rho \|f_h''\| \mathbb{E}(1(X=0)R) = \delta^3 \rho \|f_h''\| \mathbb{E}(R|X=0) = \delta\rho \frac{\mathbb{E}\bar{I}^2}{2\mathbb{E}\bar{I}}.$$

The first equality follows from $\mathbb{P}(X=0) = (1-\rho)$ [1, Section X.3]. To justify the second equality, recall from our discussion below (2.2) that in steady state, the workload cycles between busy and idle periods with lengths distributed as $\bar{B}$ and $\bar{I}$, respectively. Thus, conditioned on $X = 0$, the distribution of the residual time R is the same as the equilibrium distribution of $\bar{I}$. To prove (3.8), our starting point is

$$\lambda \mathbb{E} f_h''(X) - \sum_{m=1}^{\infty} \mathbb{E}\big(1(\tau_m \le 1) f_h''(X(\tau_m-))\big)$$

$$= \sum_{m=1}^{\infty} \mathbb{E}\Big(1(\tau_m \le 1)\Big(\lambda \int_0^{U_m} f_h''\big((X(\tau_m-) + \delta S_m - \delta u)^+\big)du - \lambda \int_0^{U_m} f_h''(X(\tau_m-))du\Big)\Big).$$

Since

$$\big|f_h''\big((X(\tau_m-) + \delta S_m - \delta u)^+\big) - f_h''\big(X(\tau_m-)\big)\big| \le \|f_h'''\|\delta(S_m + u),$$

it follows that

$$\sum_{m=1}^{\infty} \mathbb{E}\Big(1(\tau_m \le 1)\Big|\lambda \int_0^{U_m} f_h''\big((X(\tau_m-) + \delta S_m - \delta u)^+\big)du - \lambda \int_0^{U_m} f_h''(X(\tau_m-))du\Big|\Big)$$
$$\le \sum_{m=1}^{\infty} \mathbb{E}\big(1(\tau_m \le 1)\lambda\delta(U_m S_m + U_m^2/2)\|f_h'''\|\big) \le \lambda\delta\Big(\mathbb{E}S + \frac{\mathbb{E}U^2}{2\mathbb{E}U}\Big)\|f_h'''\|,$$

where in the final inequality we used the fact that $\mathbb{E}\big(\sum_{m=1}^{\infty} 1(\tau_m \le 1)\big) = \lambda$; e.g., [11, Lemma 6.1]. Since $\|f_h'''\| \le 4\|h'\|/\sigma^2$ and $\sigma^2 = \delta^2\lambda\mathbb{E}(S - \rho U)^2$, the bound in (3.8) follows.

The bound in (3.9) is argued similarly. Namely,

$$\sum_{m=1}^{\infty} \mathbb{E}\Big(1(\tau_m \le 1)\int_0^{\delta(S-\rho U)} (\delta(S - \rho U) - v)\int_0^v f_h'''(X(\tau_m-) + u)dudv\Big)$$
$$\le \delta^3\lambda\mathbb{E}\,|S - \rho U|^3\,\|f_h'''\| \le 4\delta\frac{\mathbb{E}\,|S - \rho U|^3}{\mathbb{E}(S - \rho U)^2}.$$

□

3.1. *Deriving the stationary equation.* For a version of Lemma 2 involving Palm calculus, see [2, Section 1.3.3] or [33, Theorem 2.1].

PROOF OF LEMMA 2. Initialize $(X(0), R(0))$ according to (X, R). We make frequent references to Lemma 1 for bounds on $f_h(x)$ and its derivatives. Since $f_h(x)$ is differentiable, the fundamental theorem of calculus yields

$$f_h(X(t) - \delta\rho R(t)) - f(X(0) - \delta\rho R(0))$$
$$= \int_0^t -\big(\delta 1(X(s) > 0) - \delta\rho\big) f_h'(X(s) - \delta\rho R(s))ds + \sum_{m=1}^{\infty} 1(\tau_m \le t)\Delta f_h(X(\tau_m) - \delta\rho R(\tau_m)).$$

We claim that $\mathbb{E}\,|f_h(X - \delta\rho R)| < \infty$, which follows from $|f_h(x)| \le \frac{1}{2}x^2\|h'\|/\theta$ and $\mathbb{E}(X - \delta\rho R)^2 < \infty$. The latter is true because $\mathbb{E}R^2 = \lambda\mathbb{E}U^3/3 < \infty$ by (2.3), and $\mathbb{E}X^2 < \infty$ since $\mathbb{E}S^3 < \infty$ [1, Theorems X.3.4 and X.2.1]. Taking expectations of both sides yields

$$0 = \mathbb{E}\int_0^t -\big(\delta 1(X(s) > 0) - \delta\rho\big) f_h'(X(s) - \delta\rho R(s))ds$$
$$+ \mathbb{E}\sum_{m=1}^{\infty} 1(\tau_m \le t)\Delta f_h(X(\tau_m) - \delta\rho R(\tau_m)).$$

By the Fubini-Tonelli theorem, we can interchange the expectation and integral because $|f_h'(x)| \le |x|\,\|h'\|/\theta$ and $\mathbb{E}X < \infty$. Assuming for now that we can also interchange the expectation and summation, we arrive at

$$0 = -\delta\mathbb{E}\big((1(X > 0) - \rho)f_h'(X - \delta\rho R)\big) + \sum_{m=1}^{\infty} \mathbb{E}\big(1(\tau_m \le 1)\Delta f_h(X(\tau_m) - \delta\rho R(\tau_m))\big).$$

Let $U_m = \tau_{m+1} - \tau_m$ be the interarrival time of the customer arriving at τ_{m+1} and let S_m be the workload brought by the customer arriving at τ_m, and observe that

$$\Delta f_h(X(\tau_m) - \delta\rho R(\tau_m)) = f_h(X(\tau_m-) + \delta S_m - \delta\rho U_m) - f_h(X(\tau_m-)). \tag{3.10}$$

Since S_m and U_m are both independent of τ_m and $X(\tau_m-)$, and $\mathbb{E}(S_m - \rho U_m) = 0$, Lemma 2 follows from using the Taylor expansion

$$f(x+y) - f(x) = yf'(x) + \frac{1}{2}y^2 f''(x) + \int_0^y (y-v)\int_0^v f'''(x+u)dudv$$

with $x = X(\tau_m-)$ and $y = \delta(S_m - \rho U_m)$. It remains to verify the interchange of the expectation and summation. Using (3.10) and the fact that $|f_h'(x)| \le |x|\,\|h'\|/\theta$, it follows that

$$\begin{aligned}
&\sum_{m=1}^{\infty} \mathbb{E}\,|1(\tau_m \le 1)\Delta f_h(X(\tau_m) - \delta\rho R(\tau_m))| \\
&\le \frac{\|h'\|}{\theta}\sum_{m=1}^{\infty} \mathbb{E}1(\tau_m \le 1)\,|\delta(S - \rho U)|\,(X(\tau_m-) + \delta S) \\
&\le \frac{\|h'\|}{\theta}\mathbb{E}\delta(S+\rho U)\sum_{m=1}^{\infty}\mathbb{E}\big(1(\tau_m \le 1)X(\tau_m-)\big) + \frac{\|h'\|}{\theta}\delta^2\mathbb{E}\big(S(S+\rho U)\big)\sum_{m=1}^{\infty}\mathbb{E}1(\tau_m \le 1).
\end{aligned}$$

To show that the right-hand side is finite, we need only show that

$$\sum_{m=1}^{\infty}\mathbb{E}\big(1(\tau_m \le 1)X(\tau_m-)\big) < \infty.$$

Let $N(1)$ be the number of arrivals on the interval $[0,1]$. Then

$$\begin{aligned}
\sum_{m=1}^{\infty}\mathbb{E}\big(1(\tau_m \le 1)X(\tau_m-)\big) &\le \mathbb{E}\Big(N(1)\sup_{0\le t\le 1} X(t)\Big) \\
&\le \mathbb{E}\big(N(1)(X(0) + N(1))\big) \\
&= \mathbb{E}(N(1)X(0)) + \mathbb{E}N^2(1) \le \sqrt{\mathbb{E}N^2(1)}\sqrt{\mathbb{E}X^2} + \mathbb{E}N^2(1) < \infty,
\end{aligned}$$

because $\mathbb{E}X^2 < \infty$ and $\mathbb{E}N^2(1) < \infty$ (since $\mathbb{E}U^2 < \infty$). □

3.2. *Stein factor bounds for the exponential distribution.*

PROOF OF LEMMA 1. We repeat the proof of [36, Lemma 4.1]. One readily checks that

$$f_h'(x) = -e^{\frac{2\theta}{\sigma^2}x}\int_x^{\infty}\frac{2}{\sigma^2}\big(\mathbb{E}h(Y) - h(y)\big)e^{-\frac{2\theta}{\sigma^2}x}dy$$

satisfies (3.2) for $x \in \mathbb{R}$ and $f_h'(0) = 0$. Since $h \in \text{Lip}(1)$, it is absolutely continuous and its derivative h' exists almost everywhere and satisfies $\|h'\| \le 1$. Differentiating both sides of (3.2) yields

$$-\theta f_h''(x) + \frac{1}{2}\sigma^2 f_h'''(x) = -h'(x) \tag{3.11}$$

for those x where $h'(x)$ exists, and, therefore,

$$f_h''(x) = -e^{\frac{2\theta}{\sigma^2}x}\int_x^{\infty}\frac{2}{\sigma^2}(-h'(y))e^{-\frac{2\theta}{\sigma^2}x}dy,$$

implying that $\|f_h''\| \le \|h'\|/\theta \le 1/\theta$. Rearranging the terms in (3.11) yields

$$\big|f_h'''(x)\big| = \left|\frac{2\theta}{\sigma^2} f_h''(x) - \frac{2}{\sigma^2} h'(x)\right| \le \frac{4\|h'\|}{\sigma^2} \le \frac{4}{\sigma^2}.$$

□

3.3. *The Palm inversion formula.*

PROOF OF LEMMA 3. Initialize $Z(0) = (X(0), R(0))$ according to its stationary distribution and note that

$$\mathbb{E} f_h''(X) = \mathbb{E}\Big(\int_0^1 f_h''(X(t))dt\Big).$$

Interchanging the expectation is justified since $h \in$ Lip(1) implies $\|f_h''\| < \infty$ by Lemma 1. Let $N(1)$ be the number of arrivals on $[0,1]$. Then

$$\int_0^1 f_h''(X(t))dt = \int_0^{\tau_1} f_h''(X(t))dt + \sum_{m=1}^{N(1)} \int_{\tau_m}^{\tau_{m+1}} f_h''(X(t))dt - \int_1^{\tau_{N(1)+1}} f_h''(X(t))dt.$$

Note that

$$\int_{\tau_m}^{\tau_{m+1}} f_h''(X(t))dt = \int_0^{U_m} f_h''\big((X(\tau_m-) + \delta S - \delta u)^+\big)du.$$

Furthermore, since $\tau_1 = R(0)$ and $\tau_{N(1)+1} = 1 + R(1)$,

$$\int_0^{\tau_1} f_h''(X(t))dt - \int_1^{\tau_{N(1)+1}} f_h''(X(t))dt = \int_0^{R(0)} f_h''(X(t))dt - \int_0^{R(1)} f_h''(X(1+t))dt,$$

and we note that the expected value of the right-hand side is zero by stationarity. □

4. The prelimit generator comparison approach. To simplify notation, in this section we assume that $\mathbb{P}(U > 0) = 1$. Generalizing the results to the case when $\mathbb{P}(U > 0) < 1$ is straightforward. We begin with an informal outline of the prelimit approach. Recall that any $z \in \mathbb{S}$ takes the form $z = (x, r)$ and define the workload generator

$$G_Z f(z) = \lim_{t \to 0} \frac{\mathbb{E}_z f(Z(t)) - f(z)}{t}, \quad z \in \mathbb{S},$$

for any function $f(z)$ for which the right-hand side is well defined. Then, provided that the value function

$$F_h(z) = \int_0^\infty \big(\mathbb{E}_z h(X(t)) - \mathbb{E}h(X)\big)dt, \quad z \in \mathbb{S},$$

exists and is sufficiently regular, it satisfies the Poisson equation

$$G_Z F_h(z) = -\delta 1(x > 0)\partial_x F_h(z) - \partial_r F_h(z) = \mathbb{E}h(X) - h(x), \quad z \in \mathbb{S}. \tag{4.1}$$

The left-hand side of (4.1) does not have jump terms like in the stationary equation in Section 3.1, because the infinitesimal drift from any state $z = (x, r)$ with $r > 0$ is deterministic. Thus, $G_Z f(z)$ does not encode the full dynamics of the workload process for arbitrary $f(z)$. However, $F_h(z)$ is special because it lies in the domain of the generator; i.e., $\mathbb{E}\big(1(\tau_m \le 1)\Delta F_h(Z(\tau_m))\big) = 0$ in the notation of Section 3.1.

Similar to (3.1), we let Y be an exponential random variable with rate $2\theta/\sigma^2$ and G_Y be the corresponding generator. The value of σ^2 we use Sections 4 and 5 is *different* from the σ^2 used in Section 3.

The left-hand side of (4.1) depends on both x and r, while the right-hand side depends only on x, suggesting, at first, that we are free to choose any r that we want. However, since $G_Y f(x)$ only depends on x, it is not obvious how to compare G_Z to G_Y. We propose setting $r = R$, where R is defined by (2.3), and taking expectations in (4.1), which results in

$$-\delta 1(x>0)\partial_x \mathbb{E}F_h(x,R) + \lambda\mathbb{E}\big(F_h(x+\delta S,U) - F_h(x,U)\big) = \mathbb{E}h(X) - h(x). \tag{4.2}$$

In order to compare the left-hand side to $G_Y f(x)$, we must first decide which $f(x)$ to use. The challenge is that (4.2) depends on both $\mathbb{E}F_h(\cdot,R)$ and $\mathbb{E}F_h(\cdot,U)$. We resolve this by establishing a relationship between the two functions, allowing us to rewrite (4.2) in terms of $\mathbb{E}F_h(\cdot,R)$. We then use Taylor expansion to compare the rewritten (4.2) to $G_Y f(x)$ with $f(x) = \mathbb{E}F_h(x,R)$. The result is Lemma 7, the main result of Section 4, which gives an expression for $\mathbb{E}h(X) - \mathbb{E}h(Y)$ in terms of the second and third derivatives of $\mathbb{E}F_h(x,R)$, also known as Stein factors. Stein factor bounds are the topic of Section 5.

The remainder of Section 4 formalizes the prelimit approach. Namely, in Section 4.1, we rigorously derive (4.2). We show how this can be done without explicitly verifying that $F_h(z)$ is well defined, by working with the finite-horizon value function instead. This technique be of independent interest for models where it is challenging to verify that $F_h(z)$ is well defined. Then, in Section 4.2 we establish the relationship between $\mathbb{E}F_h(\cdot,R)$ and $\mathbb{E}F_h(\cdot,U)$ and use it to compare G_Z and G_Y via Taylor expansion, resulting in Lemma 7.

4.1. *The Poisson equation..* To use $F_h(z)$ we first need to verify that it is well defined, i.e.,

$$\int_0^\infty |\mathbb{E}_z h(X(t)) - \mathbb{E}h(X)|\,dt < \infty, \quad z \in \mathbb{S}. \tag{4.3}$$

When U and S have finite $p+1$ moments, it is shown by [19] that $|\mathbb{E}_z X(t) - \mathbb{E}X|$ decays at a rate of $1/t^{p-1}$ for a class of queueing-network models much more general than the $G/G/1$ system, but we wish to avoid using their complex machinery.

Another way to verify (4.3) is by noticing that

$$\int_0^\infty |\mathbb{E}_z h(X(t)) - \mathbb{E}h(X)|\,dt \le \int_0^\infty \mathbb{E}\big(|\mathbb{E}_z h(X(t)) - \mathbb{E}_Z h(X(t))|\big)dt,$$

where the outer expectation is taken with respect to the stationary distribution of Z. To bound the right-hand side we can couple two workload processes, one with initial condition z and one with Z, and bound the expected coupling time in terms of z and Z. Constructing such a coupling is complicated by the fact that z and Z may differ both in the initial workload $X(0)$ and residual time $R(0)$.

In this paper we propose an alternative approach to arrive at (4.2) that bypasses the need to verify (4.3) directly, and involves using the M-horizon value function

$$F_h^M(z) = \int_0^M \big(\mathbb{E}_z h(X(t)) - \mathbb{E}h(X)\big)dt, \quad z \in \mathbb{S}.$$

Our starting point is the following proposition, which is proved in Appendix A.

PROPOSITION 1. *For any $h \in \mathrm{Lip}(1)$ and almost all $M > 0$,*

$$-\delta\mathbb{E}\partial_x F_h^M(x,R) + \lambda\mathbb{E}\big(F_h^M(x+\delta S,U) - F_h^M(x,U)\big) = \mathbb{E}\big(\mathbb{E}_{x,R} h(X(M)) - h(x)\big). \tag{4.4}$$

We wish to take $M \to \infty$ in (4.4) and recover (4.2). However, since we do not assume that $F_h(z)$ is well defined, we first need to specify what we mean by both $\partial_x \mathbb{E} F_h(x,R)$ and $\mathbb{E}\big(F_h(x+\delta S, U) - F_h(x,U)\big)$. We define

$$F_h(x+\epsilon, r) - F_h(x,r) = \int_0^\infty \big(\mathbb{E}_{x+\epsilon,r} h(X(t)) - \mathbb{E}_{x,r} h(X(t))\big)dt, \quad x \geq 0,\ \epsilon, r > 0. \tag{4.5}$$

To argue that this quantity is well defined, we now introduce a synchronous coupling of the workload process. This coupling also plays a central role in Section 5, and is far simpler than the coupling described following (4.3) because the coupled workloads differ only in the initial workload, but not the initial residual time.

Given $\epsilon > 0$, let $\{Z^{(\epsilon)}(t) = (X^{(\epsilon)}(t), R(t)) : t \geq 0\}$ be a coupling of $\{Z(t) = (X(t), R(t)) : t \geq 0\}$ with initial condition $X^{(\epsilon)}(0) = X(0) + \epsilon$. Both systems share the same arrival process, and the service time of each arriving customer is identical in both systems. Similar to B_0, we define $B_0^{(\epsilon)} = \inf\{t \geq 0 : X^{(\epsilon)}(t) = 0\}$. It follows that for every sample path,

$$\frac{\partial}{\partial t}\big(X^{(\epsilon)}(t) - X(t)\big) = -\delta 1\big(X(t) = 0,\ t \in [0, B_0^{(\epsilon)}]\big),$$

$$Z^{(\epsilon)}(t) = Z(t) \text{ for } t > B_0^{(\epsilon)}. \tag{4.6}$$

We adopt the convention that $\mathbb{E}_{x,r}(\cdot)$ is the expected value conditional on $Z(0) = (x,r)$, even if the quantity inside the expectation is a function of $Z^{(\epsilon)}(t)$.

Considering (4.5), our synchronous coupling yields

$$\begin{aligned} F_h(x+\epsilon, r) - F_h(x,r) &= \int_0^\infty \big(\mathbb{E}_{x+\epsilon,r} h(X(t)) - \mathbb{E}_{x,r} h(X(t))\big)dt \\ &= \mathbb{E}_{x,r} \int_0^{B_0^{(\epsilon)}} \big(h(X^{(\epsilon)}(t)) - h(X(t))\big)dt. \end{aligned} \tag{4.7}$$

The right-hand side is well defined because $\big|h(X^{(\epsilon)}(t)) - h(X(t))\big| \leq \epsilon \|h'\|$ and $\mathbb{E}_{x,r} B_0^{(\epsilon)} = \mathbb{E}_{x+\epsilon,r} B_0 < \infty$ by (2.4). A similar line of reasoning yields the following two lemmas. The detailed proofs are found in Appendix B.

LEMMA 4. *Let $T \geq 0$ be any random variable and define $\partial_x \mathbb{E} F_h(x,T)$ by*

$$\partial_x \mathbb{E} F_h(x,T) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}\big(F_h(x+\epsilon, T) - F_h(x,T)\big), \quad x \geq 0, \tag{4.8}$$

with the convention that $\partial_x \mathbb{E} F_h(x,T) = \partial_x F_h(x,r)$ when $T = r$ is deterministic. Then for any $h \in \mathrm{Lip}(1)$,

$$\partial_x \mathbb{E} F_h(x,T) = \mathbb{E}\Big(\mathbb{E}_{x,T} \int_0^{B_0} h'(X(t))dt\Big), \quad x \geq 0. \tag{4.9}$$

In the special case that $T = r$ is deterministic, (4.9) yields an expression for $\partial_x F_h(x,r)$, which implies, in particular, that

$$\partial_x \mathbb{E} F_h(x,T) = \mathbb{E} \partial_x F_h(x,T), \quad x \geq 0. \tag{4.10}$$

LEMMA 5. *For any $h \in \mathrm{Lip}(1)$ and $x \geq 0$,*

$$\lim_{M \to \infty} \mathbb{E}\big(\mathbb{E}_{x,R} h(X(M))\big) = \mathbb{E} h(X), \tag{4.11}$$

$$\lim_{M \to \infty} \mathbb{E} \partial_x F_h^M(x,R) = \mathbb{E} \partial_x F_h(x,R), \tag{4.12}$$

$$\lim_{M \to \infty} \mathbb{E}\big(F_h^M(x+\delta S, U) - F_h^M(x,U)\big) = \mathbb{E}\big(F_h(x+\delta S, U) - F_h(x,U)\big). \tag{4.13}$$

Applying Lemma 5 to take $M \to \infty$ in (4.4) of Proposition 1, using the fact that $\partial_x \mathbb{E} F_h(x,R) = \mathbb{E}\partial_x F_h(x,R)$ from Lemma 4, and noting from (4.9) that $\partial_x \mathbb{E} F_h(0,R) = 0$, we arrive at

$$-\delta \partial_x \mathbb{E} F_h(x,R) + \lambda \mathbb{E}\big(F_h(x+\delta S, U) - F_h(x,U)\big) = \mathbb{E}h(X) - h(x), \quad x \geq 0. \tag{4.14}$$

In the following section, we replace $\mathbb{E}\big(F_h(x+\delta S, U) - F_h(x,U)\big)$ by a term where the expectation is taken over R instead of U. We then perform a Taylor expansion to compare the left-hand side with G_Y.

4.2. *Taylor expansion..* Let S' be an independent copy of S and introduce the random variable

$$J(x,r) = -(x \wedge \delta r) + \delta S', \quad (x,r) \in \mathbb{S}.$$

We present the following lemma, which is proved in Appendix C.

LEMMA 6. *For any $(x,r) \in \mathbb{S}$, $s \geq 0$, and $h \in \text{Lip}(1)$,*

$$F_h(x+\delta s, r) - F_h(x,r) = \mathbb{E}\big(F_h(x+\delta s + J(x,r), U) - F_h(x+J(x,r),U)\big) + \epsilon(x,r,s),$$

where U on the right-hand side is independent of S' and, therefore, $J(x,r)$, and

$$\begin{aligned}\epsilon(x,r,s) = \mathbb{E}^{S'}\Big(&\int_{-x\wedge(\delta r)}^{-(x+\delta s)\wedge(\delta r)} \partial_x \mathbb{E}^U F_h(x+\delta s + v + \delta S', U) dv\Big) \\ &+ \int_0^r \Big(h\big((x+\delta s - \delta t)^+\big) - h\big((x-\delta t)^+\big)\Big) dt,\end{aligned}$$

where we recall that $\mathbb{E}^U(\cdot)$ and $\mathbb{E}^{S'}(\cdot)$ denote expectations with respect to U only and S' only, respectively.

To simplify notation, we define

$$\bar{F}_h'(x) = \partial_x \mathbb{E} F_h(x,R). \tag{4.15}$$

Replacing x by $x + J(x,R)$ in the Poisson equation (4.14) and taking expectations yields

$$\begin{aligned}&\mathbb{E}h(X) - \mathbb{E}h(x+J(x,R')) \\ &= -\delta \mathbb{E}\bar{F}_h'(x+J(x,R)) + \lambda \mathbb{E}\big(F_h(x+\delta S, R) - F_h(x,R)\big) - \mathbb{E}\big(\epsilon(x,R,S)\big).\end{aligned} \tag{4.16}$$

We emphasize that $\mathbb{E}\bar{F}_h'(x+J(x,R))$ is actually $\mathbb{E}^R \partial_x \mathbb{E}^{R'} F_h'(x+J(x,R),R')$, where R and R' are independent copies. We now perform Taylor expansion on the right-hand side of (4.16). The following lemma presents the Taylor expansion of (4.16), and is proved in Appendix C.1.

LEMMA 7. *Let $\theta = \delta^2$, $\sigma^2 = \delta^2 \lambda \mathbb{E}(U-S)^2$, and let Y be exponentially distributed with rate $2\theta/\sigma^2$. Fix $h \in \text{Lip}(1)$ and assume that $\bar{F}_h''(x)$ and $\bar{F}_h'''(x)$ exist for all $x \geq 0$, and that $\mathbb{E}\left|F_h'(Y)\right|, \mathbb{E}\left|F_h''(Y)\right| < \infty$. Then*

$$\begin{aligned}\mathbb{E}h(X) - \mathbb{E}h(Y) = &\ \mathbb{E}\big(h(Y+J(Y,R)) - h(Y)\big) - \mathbb{E}\big(\epsilon(Y,R,S)\big) \\ &- \delta \mathbb{E}\Big(1(\delta R \geq Y)\big(\bar{F}_h'(\delta S) - \bar{F}_h'(Y) - \delta(S-R)\bar{F}_h''(Y)\big)\Big) \\ &- \delta \mathbb{E}\Big(1(\delta R < Y) \int_0^{\delta(S-R)} \int_0^v \bar{F}_h'''(Y+u) du dv\Big) \\ &+ \lambda \mathbb{E} \int_0^{\delta S} (\delta S - v) \int_0^v \bar{F}_h'''(Y+u) du dv\end{aligned}$$

In the next section, we verify the differentiability of $\bar{F}_h'(x)$ and bound its derivatives in order to bound the right-hand side of the expression for $\mathbb{E}h(X) - \mathbb{E}h(Y)$ in Lemma 7.

5. Stein factor bounds for the $G/G/1$ workload process. This section is focused on bounding $\bar{F}_h''(x)$ and $\bar{F}_h'''(x)$ with the ultimate goal of proving the following theorem. Explicit expressions for all constants can be recovered from the proof of the theorem at the end of this section. Recall $\mathcal{M}_3$ defined in (2.5).

THEOREM 2. *Let Y be exponentially distributed with rate $2\theta/\sigma^2$, where $\theta = \delta^2$, $\sigma^2 = \delta^2 \lambda \mathbb{E}(U-S)^2$, and $\delta = 1-\rho$. Suppose that U has a density $G'(x)$ and hazard rate $\eta(x) = G'(x)/(1-G(x))$. Define $\underline{\eta} = \inf\{\eta(x) : x \geq 0\}$ and $\overline{\eta} = \sup\{\eta(x) : x \geq 0\}$.*

(a) *If $\eta(x)$ is nonincreasing and $\overline{\eta} = \eta(0) < \infty$, then*

$$|\mathbb{E}X - \mathbb{E}Y| \leq (1-\rho)\overline{C}_3$$
$$\sup_{h \in \mathcal{M}_3} |\mathbb{E}h(X) - \mathbb{E}h(Y)| \leq (1-\rho)\overline{C}_3' + (1-\rho)^2 \overline{C}_4',$$

where $\overline{C}_3$ and $\overline{C}_3'$ are constants that depend on $\overline{\eta}$ and the first three moments of U and S. Similarly, $\overline{C}_4'$ depends on $\overline{\eta}$, the first three moments of U and first four moments of S.

(b) *If $\overline{\eta} < \infty$ and $\underline{\eta} > 0$, then*

$$|\mathbb{E}X - \mathbb{E}Y| \leq (1-\rho)\underline{C}_3$$
$$\sup_{h \in \mathcal{M}_3} |\mathbb{E}h(X) - \mathbb{E}h(Y)| \leq (1-\rho)\underline{C}_3' + (1-\rho)^2 \underline{C}_4',$$

where $\underline{C}_3$ and $\underline{C}_3'$ are constants that depend on $\overline{\eta}, \underline{\eta}$ and the first three moments of U and S. Similarly, $\underline{C}_4'$ depends on $\overline{\eta}, \underline{\eta}$, the first three moments of U and first four moments of S.

We make a few comments before moving on. Case (a) covers certain heavy-tailed interarrival distributions, like when $G'(x)$ decays polynomially in x as $x \to \infty$. Case (b) implies that the interarrival time distribution has light tails. The assumption that the density of U exists and the assumptions on the hazard rate are all made to simplify the analysis. Section 5.2.1 can accommodate other forms of hazard rates not covered by cases (a) and (b) at the expense of added complexity. Extending our results to cases when U has point masses would require more effort. We elaborate on this following the statement of Lemma 12 in Section 5.1.

Comparing Theorems 1 and 2, the former only requires $h \in \text{Lip}(1)$, whereas the latter assumes $h \in \mathcal{M}_3$, but we can combine both theorems for an even better result as follows. Let W denote the steady-state customer waiting time, let $V = X/\delta$ be the unscaled workload, and to avoid confusion let Y_1 and Y_2 denote the exponential random variables appearing in Theorems 1 and 2, respectively. It is well known that

$$\begin{aligned}\frac{\mathbb{E}\bar{I}^2}{2\mathbb{E}\bar{I}} = \frac{\mathbb{E}(S-U)^2}{2\mathbb{E}(U-S)} - \mathbb{E}W &= \frac{\mathbb{E}(S-U)^2}{2\mathbb{E}(U-S)} - \rho\frac{\mathbb{E}S^2}{2\mathbb{E}S} - \rho\mathbb{E}V \\ &= \frac{1}{1-\rho}(\mathbb{E}Y_2 - \mathbb{E}X) - \rho\frac{\mathbb{E}S^2}{2\mathbb{E}S} + \mathbb{E}X\end{aligned}$$

where the first and second equalities are due to (2.5) of Chapter X.2 and Corollary X.3.5 of [1], respectively, and the final equality follows from $\mathbb{E}(U-S) = (1/\lambda)(1-\rho)$. Under the assumptions of case (a), Theorem 2 yields

$$\frac{\mathbb{E}\bar{I}^2}{2\mathbb{E}\bar{I}} \leq \overline{C}_3 + \mathbb{E}X. \tag{5.1}$$

To bound $\mathbb{E}X$, we can either exploit the relationship between $\mathbb{E}V$ and $\mathbb{E}W$ together with one of the many bounds on $\mathbb{E}W$ listed in [20] that depend on the first two moments of U and S, or we could use the cruder bound of $\mathbb{E}X \le \mathbb{E}Y + (1-\rho)\overline{C}_3$ due to Theorem 2. In either case, combining (5.1) with the upper bound of Theorem 1 yields a bound on $\sup_{h\in \text{Lip}(1)} |\mathbb{E}h(X) - \mathbb{E}h(Y_1)|$ in terms of $\overline{\eta}$ and the first three moments of S and U alone.

Lastly, we note that in case (b), the idle period $\bar{I}$ is trivially upper bounded by an exponential random variable with rate $\underline{\eta}$. Thus, the expected equilibrium idle period length $\mathbb{E}\bar{I}^2/(2\mathbb{E}\bar{I})$ is also bounded by $1/\underline{\eta}$. Nevertheless, we include case (b) to show that it can be covered by our analysis.

We prove Theorem 2 after first introducing several intermediate results. The following two lemmas contain the Stein factor bounds for the workload process. We split the bounds into two lemmas because their proofs are very different. Lemma 8 is proved in Section 5.1 while Lemma 9 is proved in Section 5.2.

LEMMA 8. *Suppose that $\overline{\eta} < \infty$. Then for any $h \in \text{Lip}(1)$ and $x \ge 0$,*

$$\big|\delta \bar{F}_h'(x)\big| \le x(1 + (\lambda + \overline{\eta})\mathbb{E}\bar{B}) \quad \text{and} \quad |\delta \partial_x \mathbb{E}F_h(x,U)| \le x(1 + 2\overline{\eta}\mathbb{E}\bar{B}). \tag{5.2}$$

Furthermore, $\bar{F}_h''(x)$ exists for any $h \in \mathcal{M}_2$, and for any $x \ge 0$,

$$\big|\delta \bar{F}_h''(x)\big| \le (1+x)\big(1 + (\lambda + \overline{\eta})\mathbb{E}\bar{B}\big) \quad \text{and} \quad \big|\delta \partial_x^2 \mathbb{E}F_h(x,U)\big| \le (1+x)(1 + 2\overline{\eta}\mathbb{E}\bar{B}). \tag{5.3}$$

LEMMA 9. *Suppose that $\overline{\eta} < \infty$. Then $\bar{F}_h'''(x)$ exists for all $x \ge 0$ and $h \in \mathcal{M}_3$. Furthermore, for any $h \in \mathcal{M}_3$,*

(a) *if $\eta(x)$ is nonincreasing, then*

$$\begin{aligned}\big|\delta^2 \bar{F}_h'''(x)\big| \le{}& \lambda\mathbb{E}\big(\delta S(1 + x + \delta S)\big)(1 + 2\overline{\eta}\mathbb{E}\bar{B}) + \mathbb{P}(\delta U > x)3\lambda\overline{\eta}\mathbb{E}\bar{B} \\ &+ \mathbb{P}(\delta U < x < \delta \bar{I} + \delta U)\lambda\overline{\eta}(1 + \overline{\eta}\mathbb{E}S)\mathbb{E}\bar{B},\end{aligned}$$

where U and $\bar{I}$ are independent. In the special case when $h(x) = x$,

$$\big|\delta^2 \bar{F}_h'''(x)\big| \le \mathbb{P}(\delta U > x)3\lambda\overline{\eta}\mathbb{E}\bar{B} + \mathbb{P}(\delta U < x < \delta \bar{I} + \delta U)\lambda\overline{\eta}(1 + \overline{\eta}\mathbb{E}S)\mathbb{E}\bar{B}.$$

(b) *if $\underline{\eta} > 0$, then for all $x \ge 0$,*

$$\begin{aligned}\big|\delta^2 \bar{F}_h'''(x)\big| \le{}& \lambda\mathbb{E}\big(\delta S(1 + x + \delta S)\big(1 + 2\overline{\eta}\mathbb{E}\bar{B}\big)\big) + \mathbb{P}(U > x/\delta)3\lambda\overline{\eta}\mathbb{E}\bar{B} \\ &+ \mathbb{E}\big(1(U < x/\delta)e^{-\underline{\eta}(x/\delta - U)}\big)\lambda\overline{\eta}\mathbb{E}\bar{B}\end{aligned}$$

and in the special case when $h(x) = x$,

$$\big|\delta^2 \bar{F}_h'''(x)\big| \le \mathbb{P}(U > x/\delta)3\lambda\overline{\eta}\mathbb{E}\bar{B} + \mathbb{E}\big(1(U < x/\delta)e^{-\underline{\eta}(x/\delta - U)}\big)\lambda\overline{\eta}\mathbb{E}\bar{B}.$$

The bounds in both Lemmas contain $\mathbb{E}\bar{B}$. The only bound on $\mathbb{E}\bar{B}$ that we know of is

$$\mathbb{E}\bar{B} \le 0.9\rho\frac{\sqrt{\text{Var}(U-S)}}{1-\rho}\exp\left(5.4\frac{\mathbb{E}|U-S|^3}{(\text{Var}(U-S))^{3/2}} + 0.8\frac{\mathbb{E}(U-S)}{\sqrt{\text{Var}(U-S)}}\right), \tag{5.4}$$

which is due to [32] and involves the first three moments of S and U. The following lemma, proved in Section 5.3, yields an alternative bound.

LEMMA 10. *Recall that $I_1, I_2, \dots$ are i.i.d. $\bar{I}$. For any $\rho < 1$,*

$$\mathbb{E}V = \lambda\frac{\mathbb{E}S^2}{2} + \lambda\mathbb{E}\bar{B}\Big[\mathbb{E}(S-U)^+ + \sum_{k=2}^{\infty}\mathbb{E}\Big(S-U-\sum_{i=1}^{k-1} I_i\Big)^+\Big]. \tag{5.5}$$

Denoting the steady-state customer waiting time distribution by W, a consequence of (5.5) is that

$$\mathbb{E}\bar{B} = \frac{\mathbb{E}W\mathbb{E}S}{\mathbb{E}(S-U)^+ + \sum_{k=2}^{\infty}\mathbb{E}\Big(S-U-\sum_{i=1}^{k-1} I_i\Big)^+} \leq \frac{\rho Var(S-U)}{2(1-\rho)\mathbb{E}(S-U)^+}. \tag{5.6}$$

REMARK 2. The bound in (5.6) is a result of the upper bound on $\mathbb{E}W$ due to [30]. Tighter bounds on $\mathbb{E}W$ have been established since then [20], and any one of them could be used instead.

We require one final auxiliary lemma, proved in Appendix D.3, which uses the Stein factor bounds from Lemmas 8 and 9 to bound the right-hand side of the Taylor expansion in Lemma 7. We recall from (2.3) that $\mathbb{E}R^k = \lambda\mathbb{E}U^{k+1}/(k+1)$.

LEMMA 11. *Let Y be exponentially distributed with rate $2\theta/\sigma^2$, where $\theta = \delta^2$, $\sigma^2 = \delta^2\lambda\mathbb{E}(U-S)^2$, and set $\nu = 2\theta/\sigma^2$. For any $h \in \mathcal{M}_3$,*

$$\mathbb{E}\left|h(Y+J(Y,R)) - h(Y)\right| \leq \delta(\mathbb{E}R + \mathbb{E}S), \tag{5.7}$$

$$\mathbb{E}\left|\epsilon(Y,R,S)\right| \leq \delta\mathbb{E}R\mathbb{E}S + \delta^2\nu\mathbb{E}R(\mathbb{E}S^2 + (\mathbb{E}S)^2)(1+2\overline{\eta}\mathbb{E}\bar{B}) \tag{5.8}$$

$$\begin{aligned}&\delta\mathbb{E}\left|1(\delta R \geq Y)\big(\bar{F}_h'(\delta S) - \bar{F}_h'(Y) - \delta(S-R)\bar{F}_h''(Y)\big)\right| \\ &\qquad \leq \delta^2\Big(\nu\big(2\mathbb{E}S\mathbb{E}R + 5\mathbb{E}R^2\big) + \mathbb{E}R^2\Big)\big(1+(\lambda+\overline{\eta})\mathbb{E}\bar{B}\big)\end{aligned} \tag{5.9}$$

Furthermore,

(a) *if $\eta(x)$ is nonincreasing, then*

$$\begin{aligned}&\lambda\Big|\mathbb{E}\int_0^{\delta S}(\delta S - v)\int_0^v \bar{F}_h'''(Y+u)dudv\Big| \\ &\qquad \leq \lambda\delta\big(\mathbb{E}S^3\mathbb{E}\big(S(1+1/\nu+\delta S)\big) + \delta\mathbb{E}S^4\big)\lambda\delta\big(1+2\overline{\eta}\mathbb{E}\bar{B}\big) \\ &\qquad + \lambda\delta\mathbb{E}S^3\big(\nu\delta 3\overline{\eta}\mathbb{E}\bar{B} + \nu\delta(\mathbb{E}U+\mathbb{E}\bar{I})\lambda\overline{\eta}(1+\overline{\eta}\mathbb{E}S)\mathbb{E}\bar{B}\big)\end{aligned} \tag{5.10}$$

$$\begin{aligned}&\delta\Big|\mathbb{E}1(\delta R < Y)\int_0^{\delta(S-R)}\int_0^v \bar{F}_h'''(Y+u)dudv\Big| \\ &\qquad \leq \delta\mathbb{E}\Big((S-R)^2\Big(\big(S'(1+1/\nu+\delta S+\delta S')\big)\lambda\delta\big(1+2\overline{\eta}\mathbb{E}\bar{B}\big) \\ &\qquad\qquad + \nu\delta 3\overline{\eta}\mathbb{E}\bar{B} + \nu\delta(\mathbb{E}U+\mathbb{E}\bar{I})\lambda\overline{\eta}(1+\overline{\eta}\mathbb{E}S)\mathbb{E}\bar{B}\Big)\Big),\end{aligned} \tag{5.11}$$

and in the special case that $h(x) = x$,

$$\begin{aligned}&\lambda\Big|\mathbb{E}\int_0^{\delta S}(\delta S - v)\int_0^v \bar{F}_h'''(Y+u)dudv\Big| \\ &\qquad \leq \lambda\delta\mathbb{E}S^3\Big(\nu\delta 3\overline{\eta}\mathbb{E}\bar{B} + \nu\delta(\mathbb{E}U+\mathbb{E}\bar{I})\lambda\overline{\eta}(1+\overline{\eta}\mathbb{E}S)\mathbb{E}\bar{B}\Big)\end{aligned} \tag{5.12}$$

(b) *if $\underline{\eta} > 0$, then for all $x \geq 0$,*

$$\lambda \mathbb{E} \int_0^{\delta S} (\delta S - v) \int_0^v \bar{F}_{\hat{h}}'''(Y+u) du dv$$
$$\leq \lambda\delta\big(\mathbb{E}S^3 \mathbb{E}\big(S(1+1/\nu+\delta S)\big) + \delta \mathbb{E}S^4\big)\lambda\delta\big(1+2\overline{\eta}\mathbb{E}\bar{B}\big)$$
$$+ \lambda\delta\mathbb{E}S^3\big(\nu\delta 3\overline{\eta}\mathbb{E}\bar{B} + \delta\nu\lambda(\overline{\eta}/\underline{\eta})\mathbb{E}\bar{B}\big) \tag{5.13}$$

$$\delta\mathbb{E}\int_0^{J(Y,R)}\int_0^v \bar{F}_{\hat{h}}'''(Y+u)dudv$$
$$\leq \delta\mathbb{E}\Big((S-R)^2\Big(\big(S'(1+1/\nu+\delta S+\delta S')\big)\lambda\delta\big(1+2\overline{\eta}\mathbb{E}\bar{B}\big)$$
$$+ \nu\delta 3\overline{\eta}\mathbb{E}\bar{B} + \delta\nu\lambda(\overline{\eta}/\underline{\eta})\mathbb{E}\bar{B}\Big)\Big) \tag{5.14}$$

and in the special case that $h(x) = x$,

$$\lambda\mathbb{E}\int_0^{\delta S}(\delta S - v)\int_0^v \bar{F}_{\hat{h}}'''(Y+u)dudv \leq \lambda\delta\mathbb{E}S^3\big(\nu\delta 3\overline{\eta}\mathbb{E}\bar{B} + \delta\nu\lambda(\overline{\eta}/\underline{\eta})\mathbb{E}\bar{B}\big) \tag{5.15}$$

REMARK 3. Note that all $\mathbb{E}\bar{B}$ terms appearing in the bounds of Lemma 11 are multiplied by $\delta = (1-\rho)$, which compensates for the $1/(1-\rho)$ term appearing in the bounds of $\mathbb{E}\bar{B}$ in (5.4) and (5.6).

PROOF OF THEOREM 2. The result follows from applying the bounds in Lemma 11 to the expression for $\mathbb{E}h(X) - \mathbb{E}h(Y)$ in Lemma 7. To bound $\mathbb{E}\bar{B}$, we use the better bound among (5.4) and (5.6), noting that the former depends only on the first three moments of U and S. □

5.1. *Second-derivative bound.* In this section we prove Lemma 8. Recall that $A(t)$ is the age of the interarrival process at time t. Our first step is the following lemma, which is proved in Appendix D.1 by differentiating $\partial_x F_h(x,r)$.

LEMMA 12. *For any $h \in \mathcal{M}_2$, any absolutely continuous random variable $T \geq 0$ with bounded density $\theta(x)$, and any $x \geq 0$,*

$$\partial_x^2 \mathbb{E}F_h(x,T) = \partial_x \mathbb{E}F_{h'}(x,T) + \frac{1}{\delta}h'(0)$$
$$+ \frac{1}{\delta}\Big(\theta(x/\delta) + \mathbb{E}\big(1(T < x/\delta)\mathbb{E}_{x,T}\eta\big(A(B_0)\big)\big)\Big)\mathbb{E}\big(\partial_x F_h(\delta S, U)\big),$$

where $\partial_x \mathbb{E}F_{h'}(x,T)$ is as in Lemma 4 but with $h'(x)$ instead of $h(x)$.

REMARK 4. In the proof of Lemma 12, the term $\lim_{\epsilon\to 0}(1/\epsilon)\mathbb{P}_{x,r}(R(B_0) < \epsilon/\delta)$ appears. Since $R(B_0) = I_0$, this quantity is the density of the idle period I_0 at zero. When U has a density, the density of I_0 at zero equals $\mathbb{E}_{x,r}\eta\big(A(B_0)\big)$, a term that plays an important role in our third-derivative bounds. Extending our results to the case that U has point masses would involve working with $\lim_{\epsilon\to 0}(1/\epsilon)\mathbb{P}_{x,r}(R(B_0) < \epsilon/\delta)$ directly.

PROOF OF LEMMA 8. For any $h \in \mathcal{M}_2$ and any random variable $T \geq 0$, Lemma 4 implies that

$$|\delta\partial_x \mathbb{E}F_h(x,T)| \leq \delta\mathbb{E}\big(\mathbb{E}_{x,T}B_0\big).$$

We claim that it suffices to show

$$|\delta\mathbb{E}(\mathbb{E}_{x,R}B_0)| \le x(1+(\lambda+\overline{\eta})\mathbb{E}\bar{B}) \quad \text{and} \quad |\delta\mathbb{E}(\mathbb{E}_{x,U}B_0)| \le x(1+2\overline{\eta}\mathbb{E}\bar{B}). \tag{5.16}$$

Note that (5.2) follows trivially from (5.16). Furthermore, since Lemma 4 implies that

$$\mathbb{E}\,|\partial_x F_h(\delta S, U)| \le \mathbb{E}\bar{B}, \tag{5.17}$$

we can apply (5.16) and (5.17) to the expression for $\bar{F}_h''(x) = \partial_x^2\mathbb{E}F_h(x,R)$ in Lemma 12, together with the fact that the density of R is bounded by λ, to conclude the first bound in (5.3). Since the density of U satisfies $G'(x) = \eta(x)(1-G(x)) \le \overline{\eta}$, a similar argument yields the second bound in (5.3).

We now prove (5.16), starting with the bound on $|\delta\mathbb{E}(\mathbb{E}_{x,R}B_0)|$. Let $\hat{h}(x) = x$, in which case Lemma 4 and the fact that $\bar{F}_{\hat{h}}'(0) = 0$ yield

$$\delta\mathbb{E}(\mathbb{E}_{x,R}B_0) = \delta\bar{F}_{\hat{h}}'(x) = \int_0^x \delta\bar{F}_{\hat{h}}''(y)dy.$$

Thanks to Lemma 12, we know that the integrand satisfies

$$\delta\bar{F}_{\hat{h}}''(x) = 1 + \Big(\lambda(1-G(x/\delta)) + \mathbb{E}\big(1(R < x/\delta)\mathbb{E}_{x,R}\eta(A(B_0))\big)\Big)\mathbb{E}\bar{B},$$

where we used the facts that the density of R is $\lambda(1-G(x))$ and that $\mathbb{E}(\partial_x F_{\hat{h}}(\delta S,U)) = \mathbb{E}\bar{B}$. Our assumption that the hazard rate is bounded yields the first bound in (5.16). Since the density of U is bounded by $\overline{\eta}$, the second bound follows by a similar argument. □

5.2. *Third-derivative bound.* In this section we prove Lemma 9. Differentiating twice both sides of the Poisson equation (4.14) yields

$$\delta\bar{F}_h'''(x) = \lambda\partial_x^2\mathbb{E}\big(F_h(x+\delta S,U) - F_h(x,U)\big) + h''(x). \tag{5.18}$$

The following lemma is proved in Appendix D.2.1. It follows directly from Lemma 12 after verifying that $\partial_x^2\mathbb{E}\big(F_h(x+\delta S,U)-F_h(x,U)\big) = \mathbb{E}^S\big(\partial_x^2\mathbb{E}^U F_h(x+\delta S,U) - \partial_x^2\mathbb{E}^U F_h(x,U)\big)$.

LEMMA 13. *Suppose that U has a bounded density. Then for any $h \in \mathcal{M}_3$ and any $x \ge 0$,*

$$\begin{aligned}\delta^2\bar{F}_h'''(x) = {}& \lambda\delta\mathbb{E}\Big(\int_0^{\delta S}\partial_x^2\mathbb{E}F_{h'}(x+y,U)dy\Big) + \lambda\mathbb{E}\big(G'(x/\delta+S) - G'(x/\delta)\big)\mathbb{E}\big(\partial_x F_h(\delta S,U)\big)\\ &+ \lambda\mathbb{E}\Big(1(x/\delta < U < x/\delta+S)\mathbb{E}_{x+\delta S,U}\eta\big(A(B_0)\big)\Big)\mathbb{E}\big(\partial_x F_h(\delta S,U)\big)\\ &+ \lambda\mathbb{E}\Big(1(U<x/\delta)\Big(\mathbb{E}_{x+\delta S,U}\eta\big(A(B_0)\big) - \mathbb{E}_{x,U}\eta\big(A(B_0)\big)\Big)\Big)\mathbb{E}\big(\partial_x F_h(\delta S,U)\big) + \delta h''(x).\end{aligned}$$

All of the terms in the expression for $\bar{F}_h'''(x)$ are straightforward to bound, with the exception of

$$\mathbb{E}\Big(1(U<x/\delta)\Big(\mathbb{E}_{x+\delta S,U}\eta\big(A(B_0)\big) - \mathbb{E}_{x,U}\eta\big(A(B_0)\big)\Big)\Big)\mathbb{E}\big(\partial_x F_h(\delta S,U)\big).$$

Naively bounding this term by $\overline{\eta}\mathbb{E}\bar{B}$ is not good enough. Sharper bounds are presented in the following lemma, which is proved in Section 5.2.1.

LEMMA 14. *Assume that $\overline{\eta} < \infty$. For any $x, s \ge 0$ and $r < x/\delta$,*

(a) *if $\eta(x)$ is nonincreasing, then*

$$\big|\mathbb{E}_{x+\delta s,r}\eta\big(A(B_0)\big) - \mathbb{E}_{x,r}\eta\big(A(B_0)\big)\big| \le \overline{\eta}(1+\overline{\eta}s)\mathbb{P}(\bar{I} > x/\delta - r).$$

(b) *if* $\underline{\eta} > 0$, *then*

$$\big|\mathbb{E}_{x+\delta s,r}\eta\big(A(B_0)\big) - \mathbb{E}_{x,r}\eta\big(A(B_0)\big)\big| \le \overline{\eta}e^{-\underline{\eta}(v-r)}.$$

PROOF OF LEMMA 9. Consider the expression of $\delta^2\bar{F}_h'''(x)$ in Lemma 13. We now bound each term there one by one. Using (5.3),

$$\lambda\mathbb{E}\Big(\int_0^{\delta S}\delta\partial_x^2\mathbb{E}F_{h'}(x+y,U)dy\Big) \le \lambda\mathbb{E}\big(\delta S(1+x+\delta S)\big)(1+2\overline{\eta}\mathbb{E}\bar{B}).$$

Next, since $G'(x) = \eta(x)\mathbb{P}(U > x) \le \overline{\eta}\mathbb{P}(U > x)$,

$$\lambda\,\big|\mathbb{E}\big(G'(x/\delta+S) - G'(x/\delta)\big)\big|\,\mathbb{E}\big(\partial_x F_h(\delta S, U)\big) \le 2\lambda\overline{\eta}\mathbb{P}(U > x/\delta)\mathbb{E}\bar{B},$$

and

$$\lambda\mathbb{E}\Big(1(x/\delta < U < x/\delta+S)\mathbb{E}_{x+\delta S,U}\eta\big(A(B_0)\big)\Big)\mathbb{E}\big(\partial_x F_h(\delta S,U)\big) \le \lambda\mathbb{P}(U > x/\delta)\overline{\eta}\mathbb{E}\bar{B}.$$

Lastly, Lemma 14 and the fact that $\|h''\| \le 1$ imply that in case (a),

$$\lambda\mathbb{E}\Big(1(U < x/\delta)\,\big|\mathbb{E}_{x+\delta S,U}\eta\big(A(B_0)\big) - \mathbb{E}_{x,U}\eta\big(A(B_0)\big)\big|\,\Big)\mathbb{E}\big(\partial_x F_h(\delta S,U)\big)$$

is bounded by

$$\begin{aligned}&\lambda\overline{\eta}(1+\overline{\eta}\mathbb{E}S)\mathbb{E}^U\big(1(U < x/\delta)\mathbb{E}^{\bar{I}}(1(\bar{I} > x/\delta - U))\big)\mathbb{E}\bar{B}\\ &= \lambda\overline{\eta}(1+\overline{\eta}\mathbb{E}S)\mathbb{P}(U < x/\delta < \bar{I}+U)\mathbb{E}\bar{B},\end{aligned}$$

where U and $\bar{I}$ are independent, and in case (b), it is bounded by

$$\lambda\overline{\eta}\mathbb{E}\big(1(U < x/\delta)e^{-\underline{\eta}(x/\delta-U)}\big)\mathbb{E}\bar{B}.$$

When $h(x) = x$, the bounds follow from the fact that $\partial_x\mathbb{E}F_{\hat{h}'}(x,U) = 0$. □

5.2.1. *The renewal process driven by idle times.* In this section we prove Lemma 14. Recall that $A(t)$ is the age of the interarrival process at time $t \ge 0$, and note that given $R(0)$, the age $A(t)$, $t \in [0, R(0))$, has no impact on the evolution of the workload process. Thus we assume, without loss of generality, that $A(0) = 0$.

Define $\ell_0 = B_0$ and $\ell_n = \ell_{n-1} + I_{n-1} + B_n$, $n \ge 1$, to be the start of the zeroth and nth idle periods of $\{Z(t) : t \ge 0\}$, respectively. Also let $T_{-1} = 0$ and $T_n = \sum_{i=0}^n I_i$, $n \ge 0$, and define

$$\{\Gamma(t) = A(\ell_n + t) : t \in [T_{n-1}, T_n), n \ge 0\}, \tag{5.19}$$

which tracks $\{A(t) : t \ge 0\}$ during idle periods of the workload process.

For $x \ge 0$ we let $v = x/\delta$ denote the unscaled workload. Recall our synchronous coupling $\{Z^{(\epsilon)}(t) : t \ge 0\}$ and its initial busy-period duration $B_0^{(x)}$, which equals the time when $\{Z(t) : t \ge 0\}$ has idled for exactly v time units; see (4.6). It follows that

$$A(B^{(x)}) = \Gamma(v), \quad x = \delta v \ge 0,$$

and, therefore, for any $(x,r) \in \mathbb{S}$ and $s \ge 0$,

$$\begin{aligned}\mathbb{E}_{x+\delta s,r}\eta(A(B)) - \mathbb{E}_{x,r}\eta(A(B)) &= \mathbb{E}_{0,r}\eta(A(B^{(x+\delta s)})) - \mathbb{E}_{0,r}\eta(A(B^{(x)}))\\ &= \mathbb{E}_{0,r}\big(\eta(\Gamma(v+s)) - \eta(\Gamma(v))\big),\end{aligned}$$

where $\mathbb{E}_{x,r}(\cdot)$ is the expectation conditioned on $Z(0)=(x,r)$. Now fix x and $r<x/\delta=v$, and define $Z(0-)=\lim_{\epsilon\downarrow 0} Z(-\epsilon)$. We claim that

$$\begin{aligned}
&\mathbb{E}_{0,r}\big(\eta(\Gamma(v+s))-\eta(\Gamma(v))\big)\\
&=\mathbb{E}\Big(\mathbb{E}\big(\eta(\Gamma(v+s))-\eta(\Gamma(v))\big|\Gamma(r)\big)\Big|Z(0)=(0,r)\Big)\\
&=\mathbb{E}\Big(\mathbb{E}\big(\eta(\Gamma(v-r+s))-\eta(\Gamma(v-r))\big|\Gamma(0)\big)\Big|Z(0-)=(0,0)\Big)\\
&=\mathbb{E}\big(\eta(\Gamma(v-r+s))-\eta(\Gamma(v-r))\big|Z(0-)=(0,0)\big).
\end{aligned}$$

The first equality follows from the tower rule and the fact that conditioned on $\Gamma(r)$, the value of $\Gamma(r+t)$, $t\geq 0$, is independent of $Z(0)$. The second equality follows, once we observe that $\Gamma(r)$ given $Z(0)=(0,r)$ has the same distribution as $\Gamma(0)$ given $Z(0-)=(0,0)$. The latter claim is true because given $Z(0)=(0,r)$, the definition of $\Gamma(t)$ in (5.19) implies that $\Gamma(r)=A(\ell_1)\stackrel{d}{=}A(\bar{B})$, because $\ell_1=B_0+I_0+B_1=r+B_1$. Similarly, given $Z(0-)=(0,0)$, i.e., a customer arrives to an empty system at $t=0$, it follows that $B_0\stackrel{d}{=}\bar{B}$ and therefore, $\Gamma(0)=A(B_0)\stackrel{d}{=}A(\bar{B})$.

Now consider any process $\{\widetilde{\Gamma}(t):t\geq 0\}$ that is equivalent in distribution to $\{\Gamma(t):t\geq 0\}$. Then

$$\begin{aligned}
&\mathbb{E}\big(\eta(\Gamma(v-r+s))-\eta(\Gamma(v-r))\big)\\
&=\mathbb{E}\big(\eta(\Gamma(v-r+s))-\eta(\widetilde{\Gamma}(v-r))\big|\Gamma(0)=\widetilde{\Gamma}(0)\big)\\
&=\mathbb{E}\big(\eta(\Gamma(v-r+s))-\eta(\widetilde{\Gamma}(v-r+s))\big|\Gamma(0)=\widetilde{\Gamma}(s)\big).
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
&\mathbb{E}_{x+\delta s,r}\eta(A(B))-\mathbb{E}_{x,r}\eta(A(B))\\
&=\mathbb{E}_{0,r}\big(\eta(\Gamma(v+s))-\eta(\Gamma(v))\big)\\
&=\mathbb{E}\Big(\eta(\Gamma(v-r+s))-\eta(\widetilde{\Gamma}(v-r+s))\Big|Z(0-)=(0,0),\Gamma(0)=\widetilde{\Gamma}(s)\Big).
\end{aligned}\tag{5.20}$$

To bound the right-hand side, we now specify the joint distribution of $\{(\widetilde{\Gamma}(t),\Gamma(t)):t\geq 0\}$ and analyze the coupling time of this process.

We first argue that $\{\Gamma(t):t\geq 0\}$ is equivalent to the continuous-time Markov process defined by the generator

$$G_\Gamma f(\gamma)=f'(\gamma)+\eta(\gamma)\big(\mathbb{E}f(A(\bar{B}))-f(\gamma)\big),\quad \gamma\geq 0.\tag{5.21}$$

By (5.19), $\{\Gamma(t):t\geq 0\}$ increases at a unit rate and jumps at times $t=T_n$, $n\geq 0$, with $\Gamma(T_n)=A(\ell_{n+1})\stackrel{d}{=}A(\bar{B})$, where ℓ_n denotes the end of the nth busy period and, consequently, the start of the nth idle period. Finally, given $t>0$, the probability that a jump occurs on the interval $(t,t+dt)$ conditioned on $\Gamma(t)$ equals

$$\frac{\mathbb{P}(\Gamma(t)<U<\Gamma(t)+dt)}{\mathbb{P}(U>\Gamma(t))}=\eta(\Gamma(t))dt+o(dt),$$

where $o(dt)\to 0$ as $dt\to 0$.

Next, we specify the joint evolution of $\{(\widetilde{\Gamma}(t),\Gamma(t)):t\geq 0\}$. Defining

$$\eta_m(x,y)=\min\{\eta(x),\eta(y)\},\quad \eta_\Delta(x,y)=\max\{\eta(x),\eta(y)\}-\min\{\eta(x),\eta(y)\},$$

we let $\{(\widetilde{\Gamma}(t), \Gamma(t)),\ t \geq 0\}$ have the same distribution as the Markov process defined by the generator

$$\begin{aligned} G_J f(\tilde{\gamma}, \gamma) = {}& \partial_{\tilde{\gamma}} f(\tilde{\gamma}, \gamma) + \partial_{\gamma} f(\tilde{\gamma}, \gamma) \\ & + \eta_m(\tilde{\gamma}, \gamma)\big(\mathbb{E} f(A(\bar{B}), A(\bar{B})) - f(\tilde{\gamma}, \gamma)\big) \\ & + \eta_\Delta(\tilde{\gamma}, \gamma) 1(\eta(\gamma) < \eta(\tilde{\gamma}))\mathbb{E}\big(f(A(\bar{B}), \gamma) - f(\tilde{\gamma}, \gamma)\big) \\ & + \eta_\Delta(\tilde{\gamma}, \gamma) 1(\eta(\gamma) > \eta(\tilde{\gamma}))\mathbb{E}\big(f(\tilde{\gamma}, A(\bar{B})) - f(\tilde{\gamma}, \gamma)\big). \end{aligned} \tag{5.22}$$

Note that the marginal law of either component of this process is equivalent to the Markov process defined in (5.21). Furthermore, when this process jumps it either couples, or only one of the components jumps. Having defined our coupling, we proceed to bound (5.20).

PROOF OF LEMMA 14. Fix $x = \delta v \geq 0$ and $r < v$, and let $\tau_C = \inf\{t \geq s : \widetilde{\Gamma}(t) = \Gamma(t)\}$, it follows that

$$\begin{aligned} &= \left|\mathbb{E}\Big(\eta(\Gamma(v-r+s)) - \eta(\widetilde{\Gamma}(v-r+s))\Big| Z(0-) = (0,0), \Gamma(0) = \widetilde{\Gamma}(s)\Big)\right| \\ &\leq \overline{\eta}\mathbb{P}\Big(\tau_C > v - r + s\Big| Z(0-) = (0,0), \Gamma(0) = \widetilde{\Gamma}(s)\Big), \end{aligned}$$

If $\underline{\eta} > 0$, then $\eta_m(\tilde{\gamma}, \gamma) \geq \underline{\eta}$, and it follows by the dynamics of (5.22) that coupling is guaranteed to happen after an exponentially distributed amount of time with rate $\underline{\eta}$. Thus, the right-hand side is bounded by $\overline{\eta} e^{-\underline{\eta}(v-r)}$.

Now assume that $\eta(x)$ is nonincreasing. Note that $\Gamma(0) = A(B_0)$ by (5.19), and that $A(B_0) \stackrel{d}{=} A(\bar{B})$ given $Z(0-) = (0,0)$. If $\widetilde{\Gamma}(s) < \Gamma(s)$, then $\eta_m(\Gamma(t), \widetilde{\Gamma}(t)) = \eta(\widetilde{\Gamma}(t))$ for all $t \geq s$ until the first jump of $\{\widetilde{\Gamma}(t) : t \geq 0\}$, at which point coupling occurs. Since $\widetilde{\Gamma}(s) = \Gamma(0) \stackrel{d}{=} A(\bar{B})$, the first jump after s happens after $\bar{I}$ amount of time, implying that the probability of no jump on $(s, v-r+s]$ is at most

$$\mathbb{P}(\bar{I} > v - r). \tag{5.23}$$

Conversely, if $\widetilde{\Gamma}(s) > \Gamma(s)$, then τ_C corresponds to the first jump time after s of $\{\Gamma(t) : t \geq 0\}$. We will shortly prove that probability that $\{\Gamma(t) : t \geq 0\}$ does not jump on $(s, v-r+s]$ is at most

$$\mathbb{P}(\bar{I} > v - r)\mathbb{E}N(s) \leq \mathbb{P}(\bar{I} > v - r)\overline{\eta}s, \tag{5.24}$$

where $N(t)$ is the number of jumps made by $\{\Gamma(t) : t \geq 0\}$ on $[0, t]$. The inequality in (5.24) is justified because our nonincreasing hazard rate assumption implies that $\eta(x) \leq \overline{\eta}$, $x \geq 0$, which further implies that $\mathbb{E}N(s) \leq \overline{\eta}s$ because $\{N(t) : t \geq 0\}$ can be dominated by a Poisson process of rate $\overline{\eta}$.

Adding (5.23) and (5.24) yields an upper bound of

$$(1 + \overline{\eta}s)\mathbb{P}(\bar{I} > v - r)$$

on the probability of not coupling on the interval $(s, v-r+s]$ when $\eta(x)$ is nonincreasing.

It remains to verify (5.24). Since $Z(0-) = (0,0)$ and $\Gamma(0) = A(B_0) \stackrel{d}{=} A(\bar{B})$, the inter-jump times I_n, $n \geq 0$, of $\{\Gamma(t) : t \geq 0\}$ are i.i.d. $\bar{I}$. To make the notation more typical, let us shift the indices of the inter-jump times forward by one; i.e., the first jump happens at I_1 instead of I_0, the second jump happens at $I_1 + I_2$ instead of $I_0 + I_1$, etc. We also let $J_0 = 0$

and J_n, $n \geq 1$, be the time of the nth jump, which satisfies $J_n = J_{n-1} + I_n$. It follows that

$$\begin{aligned}\mathbb{P}(N(v-r+s) - N(s) = 0) &= \mathbb{P}(I_{N(s)+1} > v - r + s - J_{N(s)}) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(I_{n+1} > v - r + s - J_n, N(s) = n).\end{aligned}$$

Since $\{N(s) = n\} = \{J_n < s, I_{n+1} > s - J_n\}$, the right-hand side equals

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{P}(I_{n+1} > v - r + s - J_n, J_n < s) &\leq \sum_{n=1}^{\infty} \mathbb{P}(I_{n+1} > v - r, J_n < s) \\ &= \mathbb{P}(\bar{I} > v - r) \sum_{n=1}^{\infty} \mathbb{P}(J_n < s) \\ &= \mathbb{P}(\bar{I} > v - r) \sum_{n=1}^{\infty} \mathbb{P}(N(s) \geq n) \\ &= \mathbb{P}(\bar{I} > v - r) \mathbb{E}N(s),\end{aligned}$$

where the first equality follows from the independence of I_{n+1} and J_n, as well as the fact that $I_{n+1} \stackrel{d}{=} \bar{I}$.

□

5.3. *The expected duration of the busy period initialized by an arrival to an empty system.*

PROOF OF LEMMA 10. We first prove (5.5). Let $\hat{h}(x) = x$, recall that $\bar{F}_{\hat{h}}(0) = 0$ due to (4.9), and consider the Poisson equation (4.14) with $h(x) = \hat{h}(x)$ evaluated at $x = 0$, which results in

$$\lambda \mathbb{E}\big(F_{\hat{h}}(\delta S, U) - F_{\hat{h}}(0, U)\big) = \delta \mathbb{E} V,$$

Using the synchronous coupling $\{Z^{(\epsilon)}(t)\}$ introduced in Section 4.1, it follows that

$$\delta \mathbb{E} V = \lambda \mathbb{E}\Big(\int_0^{B^{(\delta S)}} \mathbb{E}_{0,U}\big(X^{(\delta S)}(t) - X(t)\big) dt \Big). \tag{5.25}$$

From (4.6) we know that the difference $X^{(\delta S)}(t) - X(t)$ decays at rate δ only during the idle periods of $\{X(t) : t \geq 0\}$. Recall that the idle and busy period durations are $I_0, I_1, \ldots$, and $B_0, B_1, B_2, \ldots$, respectively, and that $B_0 = 0$ because $X(0) = 0$. It follows that for all times t corresponding to the busy period B_k, $k \geq 1$,

$$X^{(\delta S)}(t) - X(t) = \delta \Big(S - \sum_{i=0}^{k-1} I_i \Big)^+.$$

Letting $\mathcal{I} = \{t \in [0, B^{(\delta S)}] : X(t) = 0\}$, it follows that

$$\begin{aligned}\int_0^{B^{(\delta S)}} \big(X^{(\delta S)}(t) - X(t)\big) dt &= \int_{\mathcal{I}} \big(X^{(\delta S)}(t) - X(t)\big) dt + \int_{[0,B^{(\delta S)}]\setminus\mathcal{I}} \big(X^{(\delta S)}(t) - X(t)\big) dt \\ &= \delta S^2/2 + \sum_{k=1}^{\infty} B_k \delta \Big(S - \sum_{i=0}^{k-1} I_i \Big)^+.\end{aligned}$$

We conclude (5.5) by combining this equation with (5.25), noting that B_k is independent of $(S - I_0 - I_1 \ldots - I_{k-1})^+$ and $B_k \stackrel{d}{=} \bar{B}$, and that $I_0 \stackrel{d}{=} U$ since $Z(0) = (0, U)$. The equality in (5.6) is true because $\mathbb{E}V = \lambda \mathbb{E}S^2/2 + \rho \mathbb{E}W$ due to Corollary X.3.5 of [1]. The inequality follows from the well-known bound in (7') of [30], which says that

$$\mathbb{E}W \leq \frac{\text{Var}(S-U)}{2\mathbb{E}(S-U)} = \frac{\rho \text{Var}(S-U)}{2(1-\rho)\mathbb{E}S}.$$

□

APPENDIX A: PROVING PROPOSITION 1

We require several auxiliary lemmas.

LEMMA 15. *For any $h \in \text{Lip}(1)$ and $M > 0$,*

$$\partial_x F_h^M(z) = \mathbb{E}_z \int_0^{B_0 \wedge M} h'(X(t))dt, \quad z = (x, r) \in \mathbb{S}.$$

PROOF OF LEMMA 15. The proof is identical to that of Lemma 4; see Appendix B. □

The next three lemmas are proved in Appendix A.1.

LEMMA 16. *For any $h \in \text{Lip}(1)$ and almost all $M > 0$,*

$$-\delta \partial_x F_h^M(z) - \partial_r F_h^M(z) = \mathbb{E}_z h(X(M)) - h(x), \quad z = (x, r) \in \mathbb{S}.$$

In particular, $\partial_r F_h^M(z)$ is well defined for $z \in \mathbb{S}$.

LEMMA 17. *For any differentiable $f : \mathbb{R}_+ \to \mathbb{R}$ with $\mathbb{E}|f(U)|, \mathbb{E}|f(R)|$, and $\mathbb{E}|f'(R)| < \infty$,*

$$\mathbb{E}f'(R) = \frac{1}{\mathbb{E}U}\big(\mathbb{E}f(U) - \lim_{\epsilon \to 0} f(\epsilon)\big).$$

LEMMA 18. *For any $h \in \text{Lip}(1)$ and $M > 0$,*

$$\lim_{\epsilon \to 0} F_h^M(x, \epsilon) = \mathbb{E}F_h^M(x + \delta S, U).$$

PROOF OF PROPOSITION 1. For almost all $M > 0$, Lemma 16 says that

$$-\delta \partial_x F_h^M(z) - \partial_r F_h^M(z) = \mathbb{E}_z h(X(M)) - h(x), \quad z \in \mathbb{S}. \tag{A.1}$$

Observe that $f(r) = F_h^M(x, r)$ satisfies the conditions of Lemma 17. Indeed, $|\mathbb{E}f(U)|, |\mathbb{E}f(R)| < \infty$ because $M < \infty$. Furthermore, $|\mathbb{E}f'(R)| < \infty$ follows from the expression for $\partial_r F_h^M(z)$ in Lemma 16, together with the observation that $\left|\partial_x F_h^M(z)\right| \leq M$, which follows from Lemma 15. Setting $r = R$ in (A.1) and taking expected values yields

$$\begin{aligned}
\mathbb{E}\big(\mathbb{E}_{z,R} h(X(M)) - h(x)\big) &= -\delta \mathbb{E}\partial_x F_h^M(x, R) - \mathbb{E}\partial_r F_h^M(x, R) \\
&= -\delta \mathbb{E}\partial_x F_h^M(x, R) - \lambda\Big(\mathbb{E}F_h^M(x, U) - \lim_{\epsilon \to 0} F_h^M(x, \epsilon)\Big) \\
&= -\delta \mathbb{E}\partial_x F_h^M(x, R) - \lambda\Big(\mathbb{E}F_h^M(x, U) - \mathbb{E}F_h^M(x + \delta S, U)\Big),
\end{aligned}$$

where the second and third equalities follow from Lemmas 17 and 18, respectively. □

A.1. Auxiliary Lemma Proofs.

PROOF OF LEMMA 16. We define $\tilde{h}(x) = h(x) - \mathbb{E}h(X)$ for convenience, in which case

$$F_h^M(z) = \int_0^M \mathbb{E}_z \tilde{h}(X(t))dt, \quad z \in \mathbb{S}.$$

Our goal is to prove that

$$-\delta\partial_x F_h^M(z) - \partial_r F_h^M(z) = \mathbb{E}_z h(X(M)) - h(x), \quad z = (x,r) \in \mathbb{S}. \tag{A.2}$$

Fix $z = (x,r) \in \mathbb{S}$ and suppose first that $x = 0$. On one hand,

$$F_h^{M+\epsilon}(0, r+\epsilon) = F_h^M(0, r+\epsilon) + \int_M^{M+\epsilon} \mathbb{E}_{0,r+\epsilon}\tilde{h}(X(t))dt,$$

and on the other,

$$F_h^{M+\epsilon}(0, r+\epsilon) = \int_0^{\epsilon} \mathbb{E}_{0,r+\epsilon}\tilde{h}(X(t))dt + \int_0^M \mathbb{E}_{0,r}\tilde{h}(X(t))dt = \epsilon\tilde{h}(0) + F_h^M(0,r),$$

Equating the two expressions and dividing both sides by ϵ yields

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(F_h^M(0,r+\epsilon) - F_h^M(0,r)\big) = \tilde{h}(0) - \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_M^{M+\epsilon} \mathbb{E}_{0,r+\epsilon}\tilde{h}(X(t))dt.$$

The left-hand side equals $\delta\partial_x F_h^M(z) + \partial_r F_h^M(z) = \partial_r F_h^M(z)$, since $\partial_x F_h^M(0,r) = 0$ by Lemma 15. Thus, to prove (A.2) when $x = 0$, it suffices to show that

$$\begin{aligned} &\frac{1}{\epsilon}\int_M^{M+\epsilon} \big(\mathbb{E}_{0,r+\epsilon}\tilde{h}(X(t)) - \mathbb{E}_{0,r}\tilde{h}(X(t))\big)dt \\ &= \frac{1}{\epsilon}\int_M^{M+\epsilon} \mathbb{E}_{0,r}\big(\tilde{h}(X(t-\epsilon)) - \tilde{h}(X(t))\big)dt \to 0 \end{aligned} \tag{A.3}$$

as $\epsilon \to 0$, which implies that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_M^{M+\epsilon} \mathbb{E}_{0,r+\epsilon}\tilde{h}(X(t))dt = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_M^{M+\epsilon} \mathbb{E}_{0,r}\tilde{h}(X(t))dt = \mathbb{E}_{0,r}\tilde{h}(X(M)).$$

Observe that $|X(t-\epsilon) - X(t)|$ is bounded by the workload processed during $[t-\epsilon, t]$, which is at most $\delta\epsilon$, plus any new work that arrives during $[t-\epsilon, t]$. Letting $A([t_1,t_2])$ denote the number of customers arriving during $[t_1, t_2]$, Wald's identity says that the expected workload to arrive during $[t_1,t_2]$ equals $\mathbb{E}S\mathbb{E}A([t_1,t_2])$. Thus, to prove (A.3), we observe that for any $h \in \text{Lip}(1)$ and for all $t \in [M, M+\epsilon]$,

$$\begin{aligned} \mathbb{E}_{0,r}\big|\tilde{h}(X(t-\epsilon)) - \tilde{h}(X(t))\big| &\le \mathbb{E}_{0,r}\,|X(t-\epsilon) - X(t)| \\ &\le \delta\epsilon + \mathbb{E}_{0,r}\big(\delta\mathbb{E}S\mathbb{E}(A([t-\epsilon,t]))\big) \\ &\le \delta\epsilon + \delta\mathbb{E}S\mathbb{E}_{0,r}\big(A([M-\epsilon, M+\epsilon])\big). \end{aligned}$$

It suffices to argue that the right-hand side goes to zero as $\epsilon \to 0$. By the dominated convergence theorem,

$$\lim_{\epsilon\to 0}\mathbb{E}_{0,r}\big(A([M-\epsilon, M+\epsilon])\big) = \mathbb{E}_{0,r}\big(A([M,M])\big),$$

which equals the expected number of arrivals at time M. The right-hand side may be nonzero if the distribution of U has point masses. However, since the number of point masses is at most countable, then $\mathbb{E}_{0,r}\big(A([M,M])\big)=0$ for all but at most countably many M. This proves (A.2) when $x=0$.

The case when $x>0$ follows similarly. We repeat the arguments, highlighting the differences. Given $z=(x,r)$, fix $\epsilon<x/\delta$. Then

$$F_h^{M+\epsilon}(x,r+\epsilon)=F_h^M(x,r+\epsilon)+\int_M^{M+\epsilon}\mathbb{E}_{x,r+\epsilon}\tilde{h}(X(t))dt$$

and

$$F_h^{M+\epsilon}(x,r+\epsilon)=\int_0^{\epsilon}\mathbb{E}_{x,r+\epsilon}\tilde{h}(X(t))dt+F_h^M(x-\delta\epsilon,r).$$

Equating both expressions, subtracting $F_h^M(x,r)$ from each side, and dividing by ϵ yields

$$\begin{aligned}&\frac{1}{\epsilon}\big(F_h^M(x,r+\epsilon)-F_h^M(x,r)\big)\\&=\frac{1}{\epsilon}\big(F_h^M(x-\delta\epsilon,r)-F_h^M(x,r)\big)+\frac{1}{\epsilon}\int_0^{\epsilon}\mathbb{E}_{x,r+\epsilon}\tilde{h}(X(t))dt-\frac{1}{\epsilon}\int_M^{M+\epsilon}\mathbb{E}_{x,r+\epsilon}\tilde{h}(X(t))dt.\end{aligned}$$

We now argue that each of the terms on the right-hand side has a well-defined limit as $\epsilon\to 0$, implying that the left-hand side converges to $\partial_x F_h^M(z)$, which is itself well defined. The first term on the right-hand side converges to $-\partial_x F_h^M(z)$, which we know exists for all $z\in\mathbb{S}$ by Lemma 15. Furthermore,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon}\mathbb{E}_{x,r+\epsilon}\tilde{h}(X(t))dt=\tilde{h}(x)\quad\text{ and }\quad\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_M^{M+\epsilon}\mathbb{E}_{x,r+\epsilon}\tilde{h}(X(t))dt=\mathbb{E}_{x,r}\tilde{h}(X(M)).$$

The first equality is straightforward because no arrival occurs during $[0,\epsilon]$, while the second equality is proved the same way as (A.3). □

PROOF OF LEMMA 17. For simplicity, we first assume that $\mathbb{P}(U>0)=1$. Initialize $R(0)\sim R$, let $\tau_0=0$ and τ_m be the time of the mth arrival, and let U_m, $m\geq 1$ be the interarrival times with $U_1=R(0)$ and U_m i.i.d. U for $m\geq 2$. Then $\tau_{m+1}=\tau_m+U_{m+1}$ for $m\geq 0$, and by isolating times when jumps occur, one can verify that for any $t>0$,

$$0=\frac{1}{t}\mathbb{E}\big(f(R(t))-f(R(0))\big)=\frac{1}{t}\mathbb{E}\int_0^t -f'(R(s))ds+\frac{1}{t}\mathbb{E}\sum_{m=1}^{\infty}1(\tau_m\leq t)\big(f(R(\tau_m))-f(0)\big)$$

Initialize $R(0)\sim R$, where R is defined by (2.3). Since $\mathbb{E}|f'(R)|<\infty$, the Fubini-Tonelli theorem says that

$$\frac{1}{t}\mathbb{E}\int_0^t -f'(R(s))ds=-\frac{1}{t}\int_0^t\mathbb{E}f'(R(0))ds=-\mathbb{E}f'(R).$$

To conclude, we argue that

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\sum_{m=1}^{\infty}1(\tau_m\leq t)\big(f(R(\tau_m))-f(0)\big)=\frac{1}{\mathbb{E}U}\big(\mathbb{E}f(U)-f(0)\big).$$

Since $\tau_m=\sum_{i=1}^m U_i$ and $R(\tau_m)=U_{m+1}$, and it follows that

$$\frac{1}{t}\mathbb{E}\sum_{m=1}^{\infty}1(\tau_m\leq t)\big(f(R(\tau_m))-f(0)\big)$$

$$= \frac{1}{t}\mathbb{E}\Big(1(U_1 \le t)\big(f(U_2) - f(0)\big)\Big) + \frac{1}{t}\mathbb{E}\sum_{m=2}^{\infty} 1(U_1 + \cdots + U_m \le t)\big(f(U_{m+1}) - f(0)\big)$$

(A.4)

$$= \frac{1}{t}\mathbb{P}(U_1 \le t)\mathbb{E}\big(f(U_2) - f(0)\big) + \frac{1}{t}\mathbb{E}\sum_{m=2}^{\infty} 1(U_1 + \cdots + U_m \le t)\big(f(U_{m+1}) - f(0)\big).$$

Since $U_1 \sim R$, we know by (2.3) that the first term converges to $(1/\mathbb{E}U)\mathbb{E}\big(f(U) - f(0)\big)$ as $t \to 0$, and it remains to show that the second term converges to zero as $t \to 0$. Note that

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}\sum_{m=2}^{\infty} 1(U_1 + \cdots U_m \le t)\big(f(U_{m+1}) - f(0)\big) \\
&= \frac{1}{t}\sum_{m=2}^{\infty} \mathbb{P}(U_1 \le t, U_1 + U_2 \le t, \ldots, U_1 + \cdots U_m \le t)\mathbb{E}\big(f(U_{m+1}) - f(0)\big) \\
&\le \mathbb{E}\big(f(U) - f(0)\big)\frac{1}{t}\sum_{m=2}^{\infty} \mathbb{P}(U_1 \le t, \ldots, U_m \le t) \\
&= \mathbb{E}\big(f(U) - f(0)\big)\frac{1}{t}\sum_{m=2}^{\infty} \mathbb{P}(U_1 \le t)\mathbb{P}(U \le t)^{m-1} \\
&= \mathbb{E}\big(f(U) - f(0)\big)\frac{1}{t}\mathbb{P}(U_1 \le t)\frac{\mathbb{P}(U \le t)}{1 - \mathbb{P}(U \le t)} \to 0 \quad \text{as } t \to 0,
\end{aligned}$$

where the first equality is by the independence of the U_m, $m \ge 1$, and the convergence to zero follows from $\mathbb{P}(U = 0) = 0$. The case $0 < \mathbb{P}(U > 0) < 1$ follows similarly. Since $f(U_{m+1}) - f(0) = 0$ in (A.4) if $U_{m+1} = 0$, we only need to consider those jump times τ_m when $U_{m+1} \neq 0$. In the last display, we would then replace $\mathbb{P}(U \le t)$ by $\mathbb{P}(U \le t, U > 0)$. □

PROOF OF LEMMA 18. Define $\tilde{h}(x) = h(x) - \mathbb{E}h(X)$ and consider first the case when $x = 0$. Then

$$\begin{aligned}
F_h^M(0, \epsilon) &= \int_0^{\epsilon} \mathbb{E}_{0,\epsilon}\tilde{h}(X(t))dt + \int_{\epsilon}^{M} \mathbb{E}_{0,\epsilon}\tilde{h}(X(t))dt \\
&= \int_0^{\epsilon} \mathbb{E}_{0,\epsilon}\tilde{h}(X(t))dt + \mathbb{E}\int_0^{M-\epsilon} \mathbb{E}_{\delta S,U}\tilde{h}(X(t))dt,
\end{aligned}$$

where the outer expectation is with respect to U and S. Taking $\epsilon \to 0$, the left-hand side converges to $\lim_{\epsilon\to 0} F_h^M(0, \epsilon)$ while the right-hand side converges to

$$\lim_{\epsilon\to 0}\mathbb{E}\int_0^{M-\epsilon} \mathbb{E}_{\delta S,U}\tilde{h}(X(t))dt = \mathbb{E}\int_0^{M} \mathbb{E}_{\delta S,U}\tilde{h}(X(t))dt - \lim_{\epsilon\to 0}\mathbb{E}\int_{M-\epsilon}^{M} \mathbb{E}_{\delta S,U}\tilde{h}(X(t))dt.$$

The first term equals $\mathbb{E}F_h^M(\delta S, U)$ while the second term is zero because $h \in \text{Lip}(1)$. Now suppose that $x > 0$ and take $\epsilon < x/\delta$. Arguing as before,

$$F_h^M(x, \epsilon) = \int_0^{\epsilon} \mathbb{E}_{x,\epsilon}\tilde{h}(X(t))dt + \mathbb{E}F_h^{M-\epsilon}(x - \delta\epsilon + \delta S, U).$$

To conclude, we use the fundamental theorem of calculus to write

$$\mathbb{E}F_h^{M-\epsilon}(x - \delta\epsilon + \delta S, U) = \mathbb{E}F_h^{M-\epsilon}(x + \delta S, U) + \mathbb{E}\int_0^{-\delta\epsilon} \partial_x F_h^{M-\epsilon}(x + v + \delta S, U)dv.$$

The second term on the right-hand side converges to zero because $\left|\partial_x F_h^{M-\epsilon}(z)\right| \leq M$ due to Lemma 15. The first term converges to $\mathbb{E}F_h^M(x+\delta S, U)$ because

$$\lim_{\epsilon\to 0}\mathbb{E}\int_0^{M-\epsilon}\mathbb{E}_{x+\delta S,U}\tilde{h}(X(t))dt = \mathbb{E}\int_0^{M}\mathbb{E}_{x+\delta S,U}\tilde{h}(X(t))dt - \lim_{\epsilon\to 0}\mathbb{E}\int_{M-\epsilon}^{M}\mathbb{E}_{x+\delta S,U}\tilde{h}(X(t))dt,$$

and the second term equals zero since $h \in \text{Lip}(1)$. □

APPENDIX B: PROOFS OF LEMMAS 4 AND 5

We recall the synchronous coupling $\{Z^{(\epsilon)}(t) : t \geq 0\}$ defined in Section 4.1.

PROOF OF LEMMA 4. First, observe that

$$\frac{1}{\epsilon}\mathbb{E}\Big(\int_0^{\infty}\big(\mathbb{E}_{x+\epsilon,T}h(X(t)) - \mathbb{E}_{x,T}h(X(t))\big)dt\Big)$$
$$= \frac{1}{\epsilon}\mathbb{E}\Big(\mathbb{E}_{x,T}\int_0^{B_0}\big(h(X^{(\epsilon)}(t)) - h(X(t))\big)dt\Big) + \frac{1}{\epsilon}\mathbb{E}\Big(\mathbb{E}_{x,T}\int_{B_0}^{B_0^{(\epsilon)}}\big(h(X^{(\epsilon)}(t)) - h(X(t))\big)dt\Big).$$

Note that $\left|h(X^{(\epsilon)}(t)) - h(X(t))\right|/\epsilon \leq \|h'\| \leq 1$ and $B_0^{(\epsilon)} \to B_0$ as $\epsilon \to 0$. Also note that for all $\epsilon < 1$,

$$\mathbb{E}_{x,T}B_0^{(\epsilon)} = \mathbb{E}_{x+\epsilon,T}B_0 \leq \mathbb{E}_{x+1,T}B_0 \leq \mathbb{E}\big(\mathbb{E}_{x+1+\delta S,U}B_0\big) < \infty,$$

where the second-last inequality follows from the fact that the busy period starting at state $(x+1, T)$ is made longer if the next arrival happens immediately. The DCT then implies that

$$\frac{1}{\epsilon}\mathbb{E}\Big(\mathbb{E}_{x,T}\int_0^{B_0}\big(h(X^{(\epsilon)}(t)) - h(X(t))\big)dt\Big) \to \mathbb{E}\Big(\mathbb{E}_{x,T}\int_0^{B_0}h'(X(t))dt\Big),$$
$$\frac{1}{\epsilon}\mathbb{E}\Big(\mathbb{E}_{x,T}\int_{B_0}^{B_0^{(\epsilon)}}\big(h(X^{(\epsilon)}(t)) - h(X(t))\big)dt\Big) \to 0.$$

□

PROOF OF LEMMA 5. Fix $h \in \text{Lip}(1)$. To avoid confusion, we write E^X, E^R, and $E^{X,R}$ to denote expectations with respect to X, R, and (X,R), respectively. We first prove (4.11). Using the tower property of conditional expectation,

$$\begin{aligned}\mathbb{E}\big(\mathbb{E}_{x,R}h(X(M))\big) - \mathbb{E}h(X) &= \mathbb{E}^R\big(\mathbb{E}_{x,R}h(X(M))\big) - \mathbb{E}^R\Big(\mathbb{E}^X\big[\mathbb{E}_{X,R}h(X(M))\big|R\big]\Big)\\ &= \mathbb{E}^R\Big(\mathbb{E}^X\big[\mathbb{E}_{x,R}h(X(M)) - \mathbb{E}_{X,R}h(X(M))\big|R\big]\Big)\\ &= \mathbb{E}^{X,R}\Big(\mathbb{E}_{x,R}h(X(M)) - \mathbb{E}_{X,R}h(X(M))\Big).\end{aligned}$$

Using our synchronous coupling defined in (4.6) and the fact that $h \in \text{Lip}(1)$, it follows that

$$|\mathbb{E}_{x,R}h(X(M)) - \mathbb{E}_{X,R}h(X(M))| \leq |x - X|\,\mathbb{P}_{x\vee X,R}\big(B_0 > M\big),$$

where the probability on the right-hand side corresponds to the probability that coupling does not occur by time M. Thus,

$$\lim_{M\to\infty}\left|\mathbb{E}\big(\mathbb{E}_{x,R}h(X(M))\big) - \mathbb{E}h(X)\right| \leq \lim_{M\to\infty}\mathbb{E}\Big(|x - X|\,\mathbb{P}_{x\vee X,R}\big(B_0 > M\big)\Big) = 0.$$

The last equality follows from the DCT because $\mathbb{E}X < \infty$, and because $\lim_{M\to\infty} \mathbb{P}_{x\vee x',r}\big(B_0 > M\big) = 0$ for any $x, x', r > 0$ by (2.4). To prove (4.12), one can reuse the arguments used to prove Lemma 4 to show that

$$\partial_x F_h^M(x,r) = \mathbb{E}_{x,r} \int_0^{B_0\wedge M} h'(X(t))dt \to \partial_x F_h(x,r) \quad \text{ as } M\to\infty \text{ for all } (x,r)\in\mathbb{S},$$

and also that $\lim_{M\to\infty} \mathbb{E}\partial_x F_h^M(x,R) = \mathbb{E}\lim_{M\to\infty} \partial_x F_h^M(x,R)$.

Lastly, we prove (4.13). Similar to the way we argued (4.7),

$$\begin{aligned} F_h^M(x+\epsilon,r) - F_h^M(x,r) &= \mathbb{E}_{x,r} \int_0^{B_0^{(\epsilon)}\wedge M} \big(h(X^{(\epsilon)}(t)) - h(X(t))\big)dt \\ &\to F_h(x+\epsilon,r) - F_h(x,r), \quad \text{ as } M\to\infty \text{ for all } (x,r)\in\mathbb{S}. \end{aligned}$$

Let $\hat{h}(x) = x$ and observe that since $h\in$ Lip(1) and $X^{(\epsilon)}(t) \geq X(t)$, then

$$\big|F_h^M(x+\epsilon,r) - F_h^M(x,r)\big| \leq \mathbb{E}_{x,r}\int_0^{B_0^{(\epsilon)}\wedge M} \big(X^{(\epsilon)}(t) - X(t)\big)dt \leq F_{\hat{h}}(x+\epsilon,r) - F_{\hat{h}}(x,r).$$

It remains to show that $\mathbb{E}\big(F_{\hat{h}}(x+\delta S,U) - F_{\hat{h}}(x,U)\big) < \infty$, because then we can use the DCT to conclude (4.13). The finiteness of this expectation follows from

$$\begin{aligned} \mathbb{E}\big(F_{\hat{h}}(x+\delta S,U) - F_{\hat{h}}(x,U)\big) &= \lim_{M\to\infty} \mathbb{E}\big(F_{\hat{h}}^M(x+\delta S,U) - F_{\hat{h}}^M(x,U)\big) \\ &= \lim_{M\to\infty} \mathbb{E}\big(\mathbb{E}_{x,R}X(M) - x\big) + \lim_{M\to\infty} \delta\mathbb{E}\partial_x F_{\hat{h}}^M(x,R), \end{aligned}$$

where the first equality is due to the monotone convergence theorem, since $F_{\hat{h}}^M(x+\epsilon,r) - F_{\hat{h}}^M(x,r)$ is increasing in M and is nonnegative for any $(x,r)\in\mathbb{S}$, and the second equality is due to (4.4). The right-hand side is finite by (4.11) and (4.12). $\square$

APPENDIX C: SECTION 4.2 PROOFS

PROOF OF LEMMA 6. Recall that $J(x,r) = -(x\wedge \delta r) + \delta S'$. It follows that

$$\begin{aligned} F_h(x+\delta s,r) - F_h(x,r) &= \int_0^\infty \big(\mathbb{E}_{x+\delta s,r}h(X(t)) - \mathbb{E}_{x,r}h(X(t))\big)dt \\ &= \int_0^r \Big(h\big((x+\delta s-\delta t)^+\big) - h\big((x-\delta t)^+\big)\Big)dt \\ &\quad + \mathbb{E}\Big(F_h\big(x+\delta s + J(x+\delta s,r),U\big) - F_h\big(x+J(x,r),U\big)\Big). \end{aligned}$$

To conclude, note that

$$\begin{aligned} &\mathbb{E}\Big(F_h\big(x+\delta s + J(x+\delta s,r),U\big) - F_h\big(x+J(x,r),U\big)\Big) \\ &= \mathbb{E}\Big(F_h\big(x+\delta s + J(x,r),U\big) - F_h\big(x+J(x,r),U\big)\Big) \\ &\quad + \mathbb{E}\Big(F_h\big(x+\delta s + J(x+\delta s,r),U\big) - F_h\big(x+\delta s+J(x,r),U\big)\Big). \end{aligned}$$

Using the fundamental theorem of calculus, together with Lemma 4, which shows that $\partial_x \mathbb{E}F_h(x+\delta S,U) = \mathbb{E}\partial_x F_h(x+\delta S,U)$, we arrive at

$$\mathbb{E}\Big(F_h\big(x+\delta s + J(x+\delta s,r),U\big) - F_h\big(x+\delta s+J(x,r),U\big)\Big)$$

$$
\begin{aligned}
&= \mathbb{E}\Big(\int_{-x\wedge(\delta r)}^{-(x+\delta s)\wedge(\delta r)} \partial_x F_h(x+\delta s+v+\delta S', U)dv \Big) \\
&= \mathbb{E}^{S'}\Big(\int_{-x\wedge(\delta r)}^{-(x+\delta s)\wedge(\delta r)} \mathbb{E}^U \partial_x F_h(x+\delta s+v+\delta S', U)dv \Big) \\
&= \mathbb{E}^{S'}\Big(\int_{-x\wedge(\delta r)}^{-(x+\delta s)\wedge(\delta r)} \partial_x \mathbb{E}^U F_h(x+\delta s+v+\delta S', U)dv \Big).
\end{aligned}
$$

Interchanging $\mathbb{E}^U$ with the integral in the second equality is justified by the Fubini-Tonelli theorem because $\mathbb{E}^{S'}\mathbb{E}^U \left|\partial_x F_h(x+\delta S', U)\right| \le \mathbb{E}^{S'}\mathbb{E}^U \mathbb{E}_{x+\delta S', U} B_0 < \infty$ for all $x \ge 0$ by Lemma 4 and (2.4).

□

C.1. Proving Lemma 7. We recall that $\bar F_h'(x) = \partial_x \mathbb{E} F_h(x, R)$ and that $\bar F_h''(x)$ and $\bar F_h'''(x)$ are assumed to exist. We recall (4.16), or

$$
\begin{aligned}
&\mathbb{E}h(X) - \mathbb{E}h(x+J(x,R')) \\
&= -\delta \mathbb{E}\bar F_h'(x+J(x,R')) + \lambda \mathbb{E}\big(F_h(x+\delta S, R') - F_h(x,R')\big) - \mathbb{E}\big(\epsilon(x,R',S)\big), \qquad \text{(C.1)}
\end{aligned}
$$

where $J(x,r) = -(x\wedge \delta r) + \delta S'$. The following lemma expands the first two terms on the right-hand side. We prove it after proving Lemma 7.

LEMMA 19. *For any $x \ge 0$,*

$$
\begin{aligned}
\bar F_h'(x+J(x,R')) &= \bar F_h'(x) + \delta(S'-R')\bar F_h''(x) + 1(\delta R' < x)\int_0^{\delta(S'-R')}\int_0^v \bar F_h'''(x+u)dudv \\
&\quad + 1(\delta R' \ge x)\big(\bar F_h'(\delta S') - \bar F_h'(x) - \delta(S'-R')\bar F_h''(x)\big)
\end{aligned}
$$

$$
\mathbb{E}\big(F_h(x+\delta S, R') - F_h(x,R')\big) = \delta \mathbb{E}S \bar F_h'(x) + \frac{1}{2}\delta^2 \mathbb{E}S^2 \bar F_h''(x) + \mathbb{E}\int_0^{\delta S}(\delta S - v)\int_0^v \bar F_h'''(x+u)dudv,
$$

PROOF OF LEMMA 7 . Recall that $\lambda \mathbb{E}S = \rho$. Combining Lemma 19 with (C.1) yields

$$
\begin{aligned}
&\mathbb{E}h(X) - \mathbb{E}h(x+J(x,R')) \\
&= -\delta\big(\bar F_h'(x) + \delta \mathbb{E}(S'-R')\bar F_h''(x)\big) + \lambda\big(\delta \mathbb{E}S \bar F_h'(x) + \frac{1}{2}\delta^2 \mathbb{E}S^2 \bar F_h''(x)\big) \\
&\quad - \delta \mathbb{E}\Big(1(\delta R' \ge x)\big(\bar F_h'(\delta S') - \bar F_h'(x) - \delta(S'-R')\bar F_h''(x)\big)\Big) \\
&\quad - \delta \mathbb{E}\Big(1(\delta R' < x)\int_0^{\delta(S'-R')}\int_0^v \bar F_h'''(x+u)dudv\Big) \\
&\quad + \lambda \mathbb{E}\int_0^{\delta S}(\delta S - v)\int_0^v \bar F_h'''(x+u)dudv.
\end{aligned}
$$

Using the facts that $\lambda \mathbb{E}S = \rho$, $\lambda \mathbb{E}U = 1$, and that $\mathbb{E}R' = \lambda \mathbb{E}U^2/2$, we see that the first line on the right-hand side equals

$$
-\delta(1-\rho)\bar F_h'(x) + \frac{1}{2}\delta^2\big(\lambda \mathbb{E}S^2 - 2\lambda \mathbb{E}U \mathbb{E}S' + \lambda \mathbb{E}U^2\big)\bar F_h''(x) = G_Y \bar F_h(x).
$$

Since $F_h'(0) = 0$ due to Lemma 4, our assumptions that $\mathbb{E}\left|F_h'(Y)\right|, \mathbb{E}\left|F_h''(Y)\right| < \infty$ and integration by parts yield $\mathbb{E}G_Y \bar F_h(Y) = 0$. □

PROOF OF LEMMA 19. The expression for $\bar F_h'(x+J(x,R'))$ follows from the facts that

$$\bar F_h'(x+J(x,R')) = 1(\delta R' \geq x)\bar F_h'(\delta S') + 1(\delta R' < x)\bar F_h'(x-\delta R'+\delta S')$$

and, for all $x > \delta R'$,

$$\bar F_h'(x-\delta R'+\delta S') = \bar F_h'(x) + \delta(S'-R')\bar F_h''(x) + \int_0^{\delta(S'-R')}\int_0^v \bar F_h'''(x+u)dudv.$$

Next, we argue that

$$\mathbb{E}\big(F_h(x+\delta S,R') - F_h(x,R')\big) = \mathbb{E}\int_0^{\delta S}\bar F_h'(x+v)dv, \tag{C.2}$$

so that the expression for $\mathbb{E}\big(F_h(x+\delta S,R') - F_h(x,R')\big)$ also follows from Taylor expansion of the integrand around x. To prove (C.2), note that

$$\begin{aligned}\mathbb{E}\big(F_h(x+\delta S,R') - F_h(x,R')\big) &= \mathbb{E}\int_0^{\delta S}\partial_x F_h(x+v,R')dv\\ &= \mathbb{E}^S\int_0^{\delta S}\mathbb{E}^{R'}\partial_x F_h(x+v,R')dv\\ &= \mathbb{E}^S\int_0^{\delta S}\partial_x \mathbb{E}F_h(x+v,R')dv\\ &= \mathbb{E}\int_0^{\delta S}\bar F_h'(x+v)dv.\end{aligned}$$

The first and second-last equalities follows from Lemma 4. Once we justify the interchange of the integral and expectation in the second equality using the Fubini-Tonelli theorem, (C.2) will follow. Let $\hat h(x) = x$. Using the form of $\partial_x F_h(x,r)$ from Lemma 4, it follows that for any $h \in \text{Lip}(1)$,

$$|\partial_x F_h(x,r)| \leq \mathbb{E}_{x,r}B_0 = \mathbb{E}_{x,r}\int_0^{B_0}\hat h'(X(t))dt = \partial_x F_{\hat h}(x,r).$$

Thus,

$$\mathbb{E}\int_0^{\delta S}\big|\partial_x F_h(x+v,R')\big|dv \leq \mathbb{E}\int_0^{\delta S}\partial_x F_{\hat h}(x+v,R')dv = \mathbb{E}\big(F_{\hat h}(x+\delta S,R') - F_{\hat h}(x,R')\big),$$

and the right-hand side is finite because the right-hand side of (4.13) in Lemma 5 is finite. □

APPENDIX D: STEIN FACTOR BOUND PROOFS

D.1. Second-order bounds. We first state and prove an auxiliary lemma. We then prove Lemma 12.

LEMMA 20. *For any $\epsilon > 0$ and $(x,r) \in \mathbb{S}$ with $r < x/\delta$,*

$$\frac{1}{\epsilon}\mathbb{P}_{x,r}(R(B_0) < \epsilon/\delta) \leq M/\delta, \tag{D.1}$$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{P}_{x,r}(R(B_0) < \epsilon/\delta) = \frac{1}{\delta}\mathbb{E}_{x,r}r(A(B_0)) \tag{D.2}$$

PROOF OF LEMMA 20. Let U_n denote the interarrival time of the nth customer, let $W_0 = V(0)$, and let $W_n = V(U_1 + \cdots + U_n)$ be the workload in the system right after the nth customer arrives, which includes the workload brought by the nth customer. Let

$$\sigma = \min\{n \geq 1 : U_n > W_{n-1}\}$$

be the number of customers served in the first busy period $[0, B_0]$. Now assuming that $Z(0) = (x, r) \in \mathbb{S}$ with $r < x/\delta$, it must be that $\sigma > 1$, because $W_0 = x/\delta$ and $U_1 = r$. Since $\{R(B_0) \leq \epsilon/\delta\} = \{U_\sigma \leq W_{\sigma-1} + \epsilon/\delta\}$, it follows that

$$\begin{aligned}\frac{1}{\epsilon}\mathbb{P}_{x,r}(R(B_0) \leq \epsilon/\delta) &= \frac{1}{\epsilon}\sum_{n=2}^{\infty}\mathbb{P}_{x,r}\big(U_n \leq W_{n-1} + \epsilon/\delta \big| \sigma = n\big)\mathbb{P}_{x,r}(\sigma = n)\\ &= \frac{1}{\epsilon}\sum_{n=2}^{\infty}\mathbb{E}_{x,r}\Big[\mathbb{P}_{x,r}\big(U_n \leq W_{n-1} + \epsilon/\delta \big| \sigma = n, W_{n-1}\big)\Big| \sigma = n\Big]\mathbb{P}_{x,r}(\sigma = n)\end{aligned}$$

To proceed, note that $\{\sigma = n\} = \{U_1 \leq W_0, \ldots, U_{n-1} \leq W_{n-2}, U_n > W_{n-1}\}$ for any $n \geq 1$, implying that for any $n \geq 2$,

$$\begin{aligned}&\mathbb{P}_{x,r}\big(U_n \leq W_{n-1} + \epsilon/\delta \big| \sigma = n, W_{n-1}\big)\\ &= \mathbb{P}\big(U_n \leq W_{n-1} + \epsilon/\delta \big| W_0 = x/\delta, U_1 = r, \sigma = n, W_{n-1}\big)\\ &= \mathbb{P}\big(U_n \leq W_{n-1} + \epsilon/\delta \big| W_0 = x/\delta, U_1 = r, U_1 \leq W_0, \ldots, U_{n-1} \leq W_{n-2}, U_n > W_{n-1}, W_{n-1}\big)\\ &= \mathbb{P}\big(U_n \leq W_{n-1} + \epsilon/\delta \big| U_n > W_{n-1}, W_{n-1}\big)\\ &= \mathbb{P}\big(U \leq W_{n-1} + \epsilon/\delta \big| U > W_{n-1}, W_{n-1}\big)\\ &= \frac{U(W_{n-1} + \epsilon/\delta) - U(W_{n-1})}{1 - U(W_{n-1})},\end{aligned}$$

and therefore

$$\begin{aligned}\frac{1}{\epsilon}\mathbb{P}_{x,r}(R(B_0) \leq \epsilon/\delta) &= \frac{1}{\epsilon}\sum_{n=1}^{\infty}\mathbb{E}_{x,r}\bigg[\frac{U(W_{n-1} + \epsilon/\delta) - U(W_{n-1})}{1 - U(W_{n-1})}\bigg| \sigma = n\bigg]\mathbb{P}_{x,r}(\sigma = n)\\ &= \frac{1}{\epsilon}\mathbb{E}_{x,r}\bigg[\frac{U(W_{\sigma-1} + \epsilon/\delta) - U(W_{\sigma-1})}{1 - U(W_{\sigma-1})}\bigg]. \qquad \text{(D.3)}\end{aligned}$$

To prove (D.1), observe that the right-hand side of (D.3) is bounded by M/δ because by the mean value theorem,

$$\frac{U(w + \epsilon/\delta) - U(w)}{1 - U(w)} = \frac{\epsilon}{\delta}\frac{U'(\xi)}{1 - U(w)} = \frac{\epsilon}{\delta}r(\xi)\frac{1 - U(\xi)}{1 - U(w)} \leq \frac{\epsilon}{\delta}M$$

for some $\xi \in [w, w + \epsilon/\delta]$, where the last inequality follows from $\xi \geq w$ and our assumption that $r(x) \leq M$. Once we observe that $W_{\sigma-1} = A(B_0)$, then (D.2) follows from taking $\epsilon \to 0$ in (D.3) and applying the dominated convergence theorem. □

PROOF OF LEMMA 12. Fix $h \in \mathcal{M}_2$, $x \geq 0$, and $\epsilon > 0$, and consider

$$\begin{aligned}&\frac{1}{\epsilon}\big(\partial_x \mathbb{E}F_h(x + \epsilon, T) - \partial_x \mathbb{E}F_{h'}(x, T)\big)\\ &= \frac{1}{\epsilon}\mathbb{E}\bigg(\mathbb{E}_{x,T}\int_0^{B_0}\big(h'(X^{(\epsilon)}(t)) - h'(X(t))\big)dt\bigg) + \frac{1}{\epsilon}\mathbb{E}\bigg(\mathbb{E}_{x,T}\int_{B_0}^{B_0^{(\epsilon)}} h'(X^{(\epsilon)}(t))dt\bigg).\end{aligned}$$

Repeating the proof of Lemma 4 yields

$$\frac{1}{\epsilon}\mathbb{E}\Big(\mathbb{E}_{x,T}\int_0^{B_0}\big(h'(X^{(\epsilon)}(t))-h'(X(t))\big)dt\Big)\to\partial_x\mathbb{E}F_{h'}(x,T)\quad \text{as } \epsilon\to 0.$$

Recall that $R(B_0)$ is the residual interarrival time at the end of the initial busy period (which also equals the length of the first idle period I_0). If $R(B_0)\geq\epsilon/\delta$, then there is no arrival during the interval $[B_0,B_0^{(\epsilon)})$. Since $X^{(\epsilon)}(t)(B_0)=\epsilon$, this implies that

$$\begin{aligned}&\frac{1}{\epsilon}\mathbb{E}\Big(\mathbb{E}_{x,T}\Big(1(R(B_0)\geq\epsilon/\delta)\int_{B_0}^{B_0^{(\epsilon)}}h'(X^{(\epsilon)}(t))dt\Big)\Big)\\&=\mathbb{E}\Big(\mathbb{P}_{x,T}\big(R(B_0)\geq\epsilon/\delta\big)\Big)\frac{1}{\epsilon}\int_0^{\epsilon/\delta}h'(\epsilon-\delta t)dt.\end{aligned}$$

As $\epsilon\to 0$, the right-hand side converges to

$$\mathbb{E}\Big(\mathbb{P}_{x,T}\big(R(B_0)>0\big)\Big)\frac{1}{\delta}h'(0)=\frac{1}{\delta}h'(0).$$

To justify the last equality, we observe that $R(B_0)=0$ would imply that an arrival occurs precisely at the instant that the workload hits zero. Since the workload process is right-continuous, this would imply that $X(B_0)>0$, which contradicts the definition of B_0. It remains to show that

$$\begin{aligned}&\frac{1}{\epsilon}\mathbb{E}\Big(\mathbb{E}_{x,T}\Big(1(R(B_0)<\epsilon/\delta)\int_{B_0}^{B_0^{(\epsilon)}}h'(X^{(\epsilon)}(t))dt\Big)\Big)\\&\to\frac{1}{\delta}\Big(\theta(x/\delta)+\mathbb{E}\big(1(T<x/\delta)\mathbb{E}_{x,T}r\big(A(B_0)\big)\big)\Big)\mathbb{E}\big(\partial_x F_h(\delta S,U)\big).\end{aligned}$$

Since $R(B_0)<\epsilon/\delta$ implies that an arrival occurs in $[B_0,B_0^{(\epsilon)})$, then

$$\begin{aligned}&\frac{1}{\epsilon}\mathbb{E}\Big(\mathbb{E}_{x,T}\Big(1(R(B_0)<\epsilon/\delta)\int_{B_0}^{B_0^{(\epsilon)}}h'(X^{(\epsilon)}(t))dt\Big)\Big)\\&=\frac{1}{\epsilon}\mathbb{E}\Big(\mathbb{E}_{x,T}\Big(1(R(B_0)<\epsilon/\delta)\int_0^{R(B_0)}h'(\epsilon-\delta t)dt\Big)\Big)\\&\quad+\frac{1}{\epsilon}\mathbb{E}\Big(\mathbb{E}_{x,T}\Big(1(R(B_0)<\epsilon/\delta)\mathbb{E}\big(\partial_x F_h(\epsilon-\delta R(B_0)+\delta S,U)\big)\Big)\Big).\end{aligned}$$

The first term on the right-hand side converges to zero as $\epsilon\to 0$. To analyze the second term, note that $R(B_0)<\epsilon/\delta$ implies that $T<x/\delta+\epsilon/\delta$, and if $x/\delta\leq T<x/\delta+\epsilon/\delta$ then $R(B_0)=T-x/\delta$. Therefore, the second term equals

$$\begin{aligned}&\frac{1}{\epsilon}\mathbb{E}\Big(1(T<x/\delta)\mathbb{E}_{x,T}\Big(1(R(B_0)<\epsilon/\delta)\mathbb{E}\big(\partial_x F_h(\epsilon-\delta R(B_0)+\delta S,U)\big)\Big)\\&+\frac{1}{\epsilon}\mathbb{E}\Big(1(x/\delta\leq T<x/\delta+\epsilon/\delta)\mathbb{E}_{x,T}\Big(\mathbb{E}\big(\partial_x F_h(\epsilon-(\delta T-x)+\delta S,U)\big)\Big)\end{aligned}$$

It is straightforward to check that $\sup_{0\leq x'\leq\epsilon}\big|\mathbb{E}\big(\partial_x F_h(x'+\delta S,U)\big)-\mathbb{E}\big(\partial_x F_h(\delta S,U)\big)\big|\to 0$ as $\epsilon\to 0$. Lemma 20 and the DCT then yield

$$\begin{aligned}&\frac{1}{\epsilon}\mathbb{E}\Big(1(T<x/\delta)\mathbb{E}_{x,T}\Big(1(R(B_0)<\epsilon/\delta)\mathbb{E}\big(\partial_x F_h(\epsilon-\delta R(B_0)+\delta S,U)\big)\Big)\\&\to\frac{1}{\delta}\mathbb{E}\big(1(T<x/\delta)\mathbb{E}_{x,T}r\big(A(B_0)\big)\big)\mathbb{E}\big(\partial_x F_h(\delta S,U)\big).\end{aligned}$$

Similarly, using the fact that $\theta(x)$ is bounded,

$$\frac{1}{\epsilon}\mathbb{E}\Big(1(x/\delta \le T < x/\delta + \epsilon/\delta)\mathbb{E}_{x,T}\Big(\mathbb{E}\big(\partial_x F_h(\epsilon - (\delta T - x) + \delta S, U)\big)\Big)$$
$$\to \frac{1}{\delta}\theta(x/\delta)\mathbb{E}\big(\partial_x F_h(\delta S, U)\big).$$

□

D.2. Third-order bounds.

D.2.1. *Proof of Lemma 13.*

LEMMA 21. *Suppose that U has a bounded density. Then for any $h \in \mathcal{M}_2$ and any $x \ge 0$,*

$$\partial_x^2\mathbb{E}\big(F_h(x+\delta S, U) - F_h(x,U)\big) = \mathbb{E}^S\big(\partial_x^2\mathbb{E}^U F_h(x+\delta S, U) - \partial_x^2\mathbb{E}^U F_h(x,U)\big).$$

PROOF OF LEMMA 13. Observe that

$$\delta\bar{F}_h'''(x) = \lambda\partial_x^2\mathbb{E}\big(F_h(x+\delta S,U) - F_h(x,U)\big) + h''(x)$$
$$= \lambda\mathbb{E}^S\big(\partial_x^2\mathbb{E}^U F_h(x+\delta S,U) - \partial_x^2\mathbb{E}^U F_h(x,U)\big) + h''(x),$$

where the first equality is due to (5.18) and the second is due to Lemma 21. Applying the expression for $\partial_x^2\mathbb{E}^U F_h(\cdot, U)$ from Lemma 12 to the right-hand side yields the result. □

PROOF OF LEMMA 21. Though we do not assume $F_h(z)$ to be well defined, note that

$$\partial_x\mathbb{E}\big(F_h(x+\delta S,U) - F_h(x,U)\big)$$
$$= \lim_{\epsilon\to 0}\Big(\frac{1}{\epsilon}\mathbb{E}\big(F_h(x+\epsilon+\delta S,U) - F_h(x+\delta S,U)\big) - \frac{1}{\epsilon}\mathbb{E}\big(F_h(x+\epsilon,U) - F_h(x,U)\big)\Big)$$
$$= \partial_x\mathbb{E}F_h(x+\delta S,U) - \partial_x\mathbb{E}F_h(x,U),$$

so that by differentiating both sides with respect to x, we arrive at

$$\partial_x^2\mathbb{E}\big(F_h(x+\delta S,U) - F_h(x,U)\big) = \partial_x^2\mathbb{E}F_h(x+\delta S,U) - \partial_x^2\mathbb{E}F_h(x,U).$$

By repeating the arguments used to prove Lemma 4, one can check that

$$\partial_x^2\mathbb{E}F_h(x+\delta S,U) = \partial_x\mathbb{E}^S\partial_x\mathbb{E}^U F_h(x+\delta S,U).$$

Similarly,

$$\partial_x\mathbb{E}^S\partial_x\mathbb{E}^U F_h(x+\delta S,U) = \mathbb{E}^S\partial_x^2\mathbb{E}^U F_h(x+\delta S,U)$$

follows from repeating the proof of Lemma 12. □

D.3. Proof of Lemma 11.

PROOF. We use the following inequalities throughout the proof, which follow from the fact that Y has density $\nu e^{-\nu y}$ and is independent of R.

$$\text{(D.4)} \quad \mathbb{P}(\delta R \ge Y) = (1 - e^{-\nu\delta R}) \le \nu\delta\mathbb{E}R$$

$$\text{(D.5)} \quad \mathbb{E}Y1(\delta R \ge Y) = \frac{1}{\nu}\big((1-e^{-\nu\delta R}) - \nu\delta R e^{-\nu\delta R}\big) \le 2\nu\delta^2\mathbb{E}R^2,$$

$$\text{(D.6)} \quad \mathbb{E}Y^2 1(\delta R \ge Y) = \frac{1}{\nu^2}\big(2(1-e^{-\nu\delta R}) - 2\nu\delta R e^{-\nu\delta R} - (\nu\delta R)^2 e^{-\nu\delta R}\big) \le 3\nu\delta^3\mathbb{E}R^3$$

We also use the facts that $U, S, S', R, \bar{I}$, and Y are independent, and that S and S' have the same distribution. We first prove (5.7)–(5.9), then (5.10)–(5.11), followed by (5.13)–(5.14).

Proof of (5.7)–(5.9).. Inequality (5.7) follows from $\|h'\| \leq 1$ and $J(x,r) = -(x \wedge \delta r) + \delta S'$. Next, we prove (5.8), we first note that the definition of $\epsilon(x,r,s)$ in Lemma 6 and the fact that $\|h'\| \leq 1$ both imply that

$$\int_0^r \big|h\big((x+\delta s-\delta t)^+\big) - h\big((x-\delta t)^+\big)\big|\,dt \leq r\delta s.$$

Furthermore,

$$\begin{aligned}
&1(\delta r > x)\mathbb{E}^{S'}\Big(\int_{-x}^{-(x+\delta s)} \partial_x \mathbb{E}^U F_h(x+\delta s+v+\delta S', U)dv\Big)\\
&\leq 1(\delta r > x)\delta s \mathbb{E}^{S'}\Big(\sup_{0\leq w\leq \delta s} \partial_x \mathbb{E}^U F_h(w+\delta S', U)\Big)\\
&\leq 1(\delta r > x)s(\delta s + \delta \mathbb{E}S)(1+2\overline{\eta}\mathbb{E}\bar{B}),
\end{aligned}$$

where in the second inequality we used (5.2) of Lemma 8. Combining the bounds and using (D.4) yields

$$\begin{aligned}
\mathbb{E}\,|\epsilon(Y,R,S)| &\leq \delta\mathbb{E}R\mathbb{E}S + \mathbb{P}(\delta R \leq Y)\delta(\mathbb{E}S^2 + (\mathbb{E}S)^2)(1+2\overline{\eta}\mathbb{E}\bar{B})\\
&\leq \delta\mathbb{E}R\mathbb{E}S + \delta^2\nu\mathbb{E}R(\mathbb{E}S^2 + (\mathbb{E}S)^2)(1+2\overline{\eta}\mathbb{E}\bar{B}).
\end{aligned}$$

Next we prove (5.9). Recall from Lemma 8 that

$$\begin{aligned}
&\big|\delta\bar{F}_h'(x)\big| \leq x(1+(\lambda+\overline{\eta})\mathbb{E}\bar{B}),\\
&\big|\delta\bar{F}_h''(x)\big| \leq (1+x)\big(1+(\lambda+\overline{\eta})\mathbb{E}\bar{B}\big), \quad x \geq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\,\big|1(\delta R \geq Y)\big(\delta\bar{F}_h'(\delta S) - \delta\bar{F}_h'(Y) - \delta(S-R)\delta\bar{F}_h''(Y)\big)\big|\\
&\leq \Big(\mathbb{E}\big(1(\delta R \geq Y)(\delta S + Y)\big) + \mathbb{E}\big(1(\delta R \geq Y)\delta(S-R)(1+Y)\big)\Big)\big(1+(\lambda+\overline{\eta})\mathbb{E}\bar{B}\big)\\
&\leq \delta^2\big(\nu\big(2\mathbb{E}S\mathbb{E}R + 5\mathbb{E}R^2\big) + \mathbb{E}R^2\big)\big(1+(\lambda+\overline{\eta})\mathbb{E}\bar{B}\big).
\end{aligned}$$

The last inequality follows from using (D.4) and (D.5) to show that

$$\begin{aligned}
&\mathbb{E}\big(1(\delta R \geq Y)(\delta S + Y)\big) \leq \delta^2\nu\mathbb{E}S\mathbb{E}R + 2\nu\delta^2\mathbb{E}R^2,\\
&\delta\mathbb{E}S\mathbb{E}\big(1(\delta R \geq Y)(1+Y)\big) \leq \delta\mathbb{E}S\big(\nu\delta\mathbb{E}R + 2\nu\delta^2\mathbb{E}R^2\big),\\
&\delta\mathbb{E}\big(1(\delta R \geq Y)R(1+Y)\big) \leq \delta\big(\nu\delta\mathbb{E}R^2 + \delta\mathbb{E}R^2\big),
\end{aligned}$$

where in the final inequality we used the fact that $\mathbb{E}\big(1(\delta R \geq Y)RY\big) \leq \delta\mathbb{E}R^2$ instead of (D.5). Using the latter would have resulted in a term involving $\mathbb{E}R^3$.

Proof of (5.10)–(5.12).. Recall from Lemma 9 that for any x, u such that $x + u \geq 0$,

$$\begin{aligned}
\big|\delta^2\bar{F}_h'''(x+u)\big| \leq\ & \lambda\delta\mathbb{E}\big(S(1+x+u+\delta S)\big)\big(1+2\overline{\eta}\mathbb{E}\bar{B}\big) + \mathbb{P}(\delta U > x+u)3\lambda\overline{\eta}\mathbb{E}\bar{B}\\
&+ \mathbb{P}(\delta U < x+u < \delta\bar{I} + \delta U)\lambda\overline{\eta}(1+\overline{\eta}\mathbb{E}S)\mathbb{E}\bar{B}.
\end{aligned} \tag{D.7}$$

We claim that for any $u > 0$,

$$\begin{aligned}
\mathbb{E}\big|\delta^2\bar{F}_h'''(Y+u)\big| \leq\ & \lambda\delta\mathbb{E}\big(S'(1+Y+u+\delta S')\big)\big(1+2\overline{\eta}\mathbb{E}\bar{B}\big) + \nu\delta 3\overline{\eta}\mathbb{E}\bar{B}\\
&+ \nu\delta(\mathbb{E}U + \mathbb{E}\bar{I})\lambda\overline{\eta}(1+\overline{\eta}\mathbb{E}S)\mathbb{E}\bar{B},
\end{aligned} \tag{D.8}$$

and for any $u \in [-\delta R, 0]$,

$$\mathbb{E}^Y\Big(1(\delta R \leq Y)\big|\delta^2 \bar{F}_h'''(Y+u)\big|\Big) \leq \lambda\delta\mathbb{E}\big(S'(1+Y+\delta S')\big)\big(1+2\overline{\eta}\mathbb{E}\bar{B}\big) + \nu\delta 3\overline{\eta}\mathbb{E}\bar{B}$$

$$+\nu\delta(\mathbb{E}U+\mathbb{E}\bar{I})\lambda\overline{\eta}(1+\overline{\eta}\mathbb{E}S)\mathbb{E}\bar{B}. \tag{D.9}$$

We now prove (5.10). Since

$$\lambda\mathbb{E}\Big(\int_0^{\delta S}(\delta S - v)\int_0^v \big|\bar{F}_h'''(Y+u)\big|\,dudv\Big) = \lambda\mathbb{E}^S\Big(\int_0^{\delta S}(\delta S - v)\int_0^v \mathbb{E}^Y\big|\bar{F}_h'''(Y+u)\big|\,dudv\Big),$$

applying (D.8) to the right-hand side and using the fact that $\mathbb{E}Y = 1/\nu$ yields (5.10). Note that (5.12) follows identically because due to Lemma 9, in the special case that $h(x) = x$,

$$\big|\delta^2\bar{F}_h'''(x+u)\big| \leq \mathbb{P}(\delta U > x+u)3\lambda\overline{\eta}\mathbb{E}\bar{B} + \mathbb{P}(\delta U < x+u < \delta\bar{I}+\delta U)\lambda\overline{\eta}(1+\overline{\eta}\mathbb{E}S)\mathbb{E}\bar{B}. \tag{D.10}$$

To prove (5.11), we note that

$$\begin{aligned}
&\delta\frac{1}{\delta^2}\mathbb{E}\Big(\int_0^{\delta(S-R)}\int_0^v \big|1(\delta R<Y)\delta^2\bar{F}_h'''(Y+u)\big|dudv\Big)\\
&= \delta\frac{1}{\delta^2}\mathbb{E}^{S,R}\Big(\int_0^{\delta(S-R)}\int_0^v \mathbb{E}^Y\big|1(\delta R<Y)\delta^2\bar{F}_h'''(Y+u)\big|dudv\Big)\\
&\leq \delta\frac{1}{\delta^2}\delta^2\mathbb{E}\Big((S-R)^2\Big(\lambda\delta\big(S'(1+Y+\delta S+\delta S')\big)\big(1+2\overline{\eta}\mathbb{E}\bar{B}\big)\\
&\quad + \nu\delta 3\overline{\eta}\mathbb{E}\bar{B} + \nu\delta(\mathbb{E}U+\mathbb{E}\bar{I})\lambda\overline{\eta}(1+\overline{\eta}\mathbb{E}S)\mathbb{E}\bar{B}\Big)\Big),
\end{aligned}$$

where the inequality follows from using both (D.8) and (D.9), together with the fact that $u \leq \delta S$. It remains to prove (D.8) and (D.9). For any $u > 0$,

$$\begin{aligned}
&\mathbb{E}\big(S'(1+Y+u+\delta S')\big) = \mathbb{E}\big(S'(1+1/\nu+u+\delta S')\big),\\
&\mathbb{P}(\delta U > Y+u) \leq \mathbb{P}(\delta U > Y) = \mathbb{E}(1-e^{-\nu\delta U}) \leq \nu\delta\mathbb{E}U,\\
&\mathbb{P}(\delta U < Y+u < \delta\bar{I}+\delta U) \leq \nu\delta(\mathbb{E}U+\mathbb{E}\bar{I}).
\end{aligned} \tag{D.11}$$

The last inequality is true because

$$\begin{aligned}
&\mathbb{P}(\delta U < Y+u < \delta\bar{I}+\delta U)\\
&= \mathbb{P}(Y \leq \delta U < Y+u < \delta\bar{I}+\delta U) + \mathbb{P}(\delta U < Y, Y+u < \delta\bar{I}+\delta U)\\
&\leq \mathbb{P}(\delta U \geq Y) + \mathbb{P}(\delta U < Y < \delta U + \delta\bar{I})\\
&= \mathbb{E}(1-e^{-\nu\delta U}) + \mathbb{E}e^{-\nu\delta U}(1-e^{-\nu\delta\bar{I}}) \leq \nu\delta(\mathbb{E}U+\mathbb{E}\bar{I}).
\end{aligned}$$

Combining (D.11) with (D.10) and using the fact that $\lambda\mathbb{E}U = 1$ proves (D.8). Finally, we prove (D.9). We claim that for any $u \in [-\delta R, 0]$,

$$\begin{aligned}
&\mathbb{E}\big(S(1+Y+u+S)\big) \leq \mathbb{E}\big(S(1+Y+S)\big)\\
&\mathbb{E}^{Y,U}\big(1(\delta R<Y)1(\delta U > Y+u)\big) \leq \nu\delta\mathbb{E}U,\\
&\mathbb{E}^{Y,U,\bar{I}}\big(1(\delta U < Y+u < \delta\bar{I}+\delta U)\big) \leq \nu\delta(\mathbb{E}U+\mathbb{E}\bar{I})\mathbb{E}\bar{B}.
\end{aligned} \tag{D.12}$$

The first inequality is immediate. The second follows from

$$\begin{aligned}
&\mathbb{E}^{Y,U}\big(1(\delta R<Y)1(\delta U > Y+u)\big)\\
&\leq \mathbb{E}^{Y,U}\big(1(\delta R < Y < \delta U+\delta R)\big) = e^{-\nu\delta R}\mathbb{E}(1-e^{-\nu\delta U}) \leq \nu\delta\mathbb{E}U
\end{aligned}$$

and the third from

$$E^{Y,U,\bar I}\big(1(\delta U < Y + u < \delta\bar I + \delta U)\big) \le E^{Y,U,\bar I}\big(1(\delta U - u < Y < \delta \bar I + \delta U - u)\big)$$
$$= \mathbb{E}\big(e^{-\nu(\delta U - u)}(1 - e^{-\nu\delta\bar I})\big) \le \nu\delta\mathbb{E}\bar I.$$

Proof of (5.13)–(5.15). We now argue that for any $u \in \mathbb{R}$,

$$\mathbb{E}^Y\big(1(\delta U < Y + u)e^{-(\underline{\eta}/\delta)(Y+u-\delta U)}\big)\big) \le \delta\nu/\underline{\eta}.$$

Applying this to the upper bound on $\big|\delta^2\bar F_h'''(x+u)\big|$ in Lemma 9, together with the bounds we already established when proving (5.10)–(5.12), we arrive at (5.13) and (5.14). For $u > 0$,

$$\mathbb{E}^Y\big(1(\delta U < Y + u)e^{-(\underline{\eta}/\delta)(Y+u-\delta U)}\big)\big) \le \mathbb{E}^Y\big(1(\delta U < Y+u)1(u < \delta U)e^{-(\underline{\eta}/\delta)(Y+u-\delta U)}\big)\big)$$
$$+ \mathbb{E}^Y\big(1(u \ge \delta U)e^{-(\underline{\eta}/\delta)Y}\big)\big).$$

Since Y has density $\nu e^{-\nu y}$,

$$\mathbb{E}^Y\big(1(u \ge \delta U)e^{-(\underline{\eta}/\delta)Y}\big)\big) \le \frac{\nu}{\nu + \underline{\eta}/\delta} \le \delta\nu/\underline{\eta},$$

and

$$\mathbb{E}^Y\big(1(\delta U < Y+u)1(u<\delta U)e^{-(\underline{\eta}/\delta)(Y+u-\delta U)}\big)\big)$$
$$= e^{-(\underline{\eta}/\delta)(u-\delta U)}1(u < \delta U)\int_{\delta U - u}^{\infty} \nu e^{-(\nu+\underline{\eta}/\delta)y}dy$$
$$= e^{-(\underline{\eta}/\delta)(u-\delta U)}1(u<\delta U)e^{-(\nu+\underline{\eta}/\delta)(\delta U - u)}\frac{\nu}{\nu+\underline{\eta}/\delta} \le \delta\nu/\underline{\eta}.$$

The case when $u \le 0$ is argued similarly. Note that (5.15) follows identically because due to Lemma 9, in the special case that $h(x) = x$,

$$\big|\delta^2\bar F_h'''(x+u)\big| \le \mathbb{P}(U > x/\delta)3\lambda\overline{\eta}\mathbb{E}\bar B + \mathbb{E}\big(1(U < x/\delta)e^{-\underline{\eta}(x/\delta - U)}\big)\big)\lambda\overline{\eta}\mathbb{E}\bar B.$$

□

REFERENCES

[1] ASMUSSEN, S. (2003). *Applied probability and queues*, second ed. *Applications of Mathematics (New York)* **51**. Springer-Verlag, New York. Stochastic Modelling and Applied Probability. MR1978607 (2004f:60001)

[2] BACCELLI, F. and BRÉMAUD, P. (2003). *Elements of queueing theory: Palm martingale calculus and stochastic recurrences*, 2nd ed. *Applications of Mathematics* **26**. Springer, Berlin.

[3] BARBOUR, A. D. (1990). Stein's method for diffusion approximations. *Probab. Theory and Related Fields* **84** 297-322. https://doi.org/10.1007/BF01197887

[4] BARBOUR, A., ROSS, N. and ZHENG, G. (2023). Stein's method, Gaussian processes and Palm measures, with applications to queueing. *The Annals of Applied Probability* **33** 3835–3871.

[5] BARBOUR, A. D. (1988). Stein's Method and Poisson Process Convergence. *Journal of Appl. Probab.* **25** 175-184.

[6] BERTSIMAS, D. and GAMARNIK, D. (2022). *Queueing theory: classical and modern methods*, 1st ed. Dynamic Ideas LLC, Belmont, MA.

[7] BESANÇON, E., DECREUSEFOND, L. and MOYAL, P. (2020). Stein's method for diffusive limits of queueing processes. *Queueing Systems* **95** 173–201.

[8] BOON, M., JANSSEN, A. and VAN LEEUWAARDEN, J. (2023). Heavy-traffic single-server queues and the transform method. *Indagationes Mathematicae* **34** 1014–1037.

[9] BRAVERMAN, A. (2022). The Prelimit Generator Comparison Approach of Stein's Method. *Stochastic Systems* **12** 181-204. https://doi.org/10.1287/stsy.2021.0085

[10] BRAVERMAN, A., DAI, J. G. and MIYAZAWA, M. (2017). Heavy traffic approximation for the stationary distribution of a generalized Jackson network: the BAR approach. *Stochastic Systems* **7** 143–196.

[11] BRAVERMAN, A., DAI, J. and MIYAZAWA, M. (2024). The BAR approach for multiclass queueing networks with SBP service policies. *Stochastic Systems*.

[12] BRAVERMAN, A. and DAI, J. G. (2017). Stein's method for steady-state diffusion approximations of $M/Ph/n+M$ systems. *Ann. of Appl. Probab.* **27** 550-581. https://doi.org/10.1214/16-AAP1211

[13] BRAVERMAN, A., DAI, J. G. and FENG, J. (2016). Stein's method for steady-state diffusion approximations: An introduction through the Erlang-A and Erlang-C models. *Stoch. Syst.* **6** 301-366.

[14] BROWN, T. C. and XIA, A. (2001). Stein's method and birth-death processes. *Ann. Probab.* **29** 1373–1403. https://doi.org/10.1214/aop/1015345606

[15] CHEN, Y. and WHITT, W. (2020). Extremal models for the GI/ GI/ K waiting-time tail-probability decay rate. *Operations Research Letters* **48** 770–776.

[16] CHEN, Y. and WHITT, W. (2021). Extremal GI/GI/1 queues given two moments: exploiting Tchebycheff systems. *Queueing Systems* **97** 101–124.

[17] CHEN, Y. and WHITT, W. (2022). Correction to: Extremal GI/GI/1 queues given two moments: exploiting Tchebycheff systems. *Queueing Systems* **102** 553–556.

[18] CHEN, Y. and WHITT, W. (2022). SET-VALUED PERFORMANCE APPROXIMATIONS FOR THE QUEUE GIVEN PARTIAL INFORMATION. *Probability in the Engineering and Informational Sciences* **36** 378–400.

[19] DAI, J. G. and MEYN, S. P. (1995). Stability and convergence of moments for multiclass queueing networks via fluid limit models. *IEEE Transactions on Automatic Control* **40** 1889–1904.

[20] DALEY, D., KREININ, A. Y. and TRENGOVE, C. (1992). Inequalities concerning the waiting time in single-server queues: A survey. *Oxford Statistical Science Series* **1** 177–177.

[21] GAST, N. (2017). Expected values estimated via mean-field approximation are 1/n-accurate. *Proceedings of the ACM on Measurement and Analysis of Computing Systems* **1** 1–26.

[22] GAUNT, R. E. and WALTON, N. (2020). Stein's method for the single server queue in heavy traffic. *Statistics & Probability Letters* **156** 108566. https://doi.org/10.1016/j.spl.2019.108566

[23] GÖTZE, F. (1991). On the rate of convergence in the multivariate CLT. *Ann. Probab.* **19** 724–739. https://doi.org/10.1214/aop/1176990448

[24] GURVICH, I. (2014). Diffusion models and steady-state approximations for exponentially ergodic Markovian queues. *Ann. Appl. Probab.* **24** 2527–2559. https://doi.org/10.1214/13-AAP984

[25] GURVICH, I., HUANG, J. and MANDELBAUM, A. (2014). Excursion-Based Universal Approximations for the Erlang-A Queue in Steady-State. *Math. Oper. Res.* **39** 325-373. https://doi.org/10.1287/moor.2013.0606

[26] GUT, A. (1974). On the moments and limit distributions of some first passage times. *The Annals of Probability* 277–308.

[27] HARRISON, J. M. and REIMAN, M. I. (1981). Reflected Brownian motion on an orthant. *Ann. Probab.* **9** 302–308. MR606992

[28] HUANG, J. and GURVICH, I. (2018). Beyond Heavy-Traffic Regimes: Universal Bounds and Controls for the Single-Server Queue. *Oper. Res.* **66** 1168-1188. https://doi.org/10.1287/opre.2017.1715

[29] KINGMAN, J. F. C. (1961). The single server queue in heavy traffic. *Mathematical Proceedings of the Cambridge Philosophical Society* **57** 902-904. https://doi.org/10.1017/S0305004100036094

[30] KINGMAN, J. F. C. (1962). On queues in heavy traffic. *J. Roy. Statist. Soc. Ser. B* **24** 383–392. MR0148146

[31] LI, J. and OU, J. (1995). Characterizing the idle-period distribution of GI/G/1 queues. *Journal of applied probability* **32** 247–255.

[32] LOULOU, R. (1978). An explicit upper bound for the mean busy period in a $GI/G/1$ queue. *Journal of Applied Probability* **15** 452–455. https://doi.org/10.2307/3213419

[33] MIYAZAWA, M. (1994). Rate conservation laws: a survey. *Queueing Systems* **15** 1-58.

[34] MIYAZAWA, M. (2015). Diffusion approximation for stationary analysis of queues and their networks: a review. *J. Oper. Res. Soc. Japan* **58** 104–148. https://doi.org/10.15807/jorsj.58.104 MR3363214

[35] MIYAZAWA, M. (2017). A unified approach for large queue asymptotics in a heterogeneous multiserver queue. *Adv. in Appl. Probab.* **49** 182–220. https://doi.org/10.1017/apr.2016.84 MR3631221

[36] PEKÖZ, E. A. and RÖLLIN, A. (2011). New rates for exponential approximation and the theorems of Rényi and Yaglom. *The Annals of Probability* **39** 587 – 608. https://doi.org/10.1214/10-AOP559

[37] ROSS, N. (2011). Fundamentals of Stein's method. *Probab. Surv.* **8** 210–293. https://doi.org/10.1214/11-PS182 MR2861132 (2012k:60079)

[38] STEIN, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Probability Theory* 583–602. University of California Press, Berkeley, Calif.

[39] Stolyar, A. L. (2015). Tightness of stationary distributions of a flexible-server system in the Halfin-Whitt asymptotic regime. *Stoch. Syst.* **5** 239–267. https://doi.org/10.1214/14-SSY139
[40] Wolff, R. W. and Wang, C.-L. (2003). Idle period approximations and bounds for the GI/G/1 queue. *Advances in Applied Probability* **35** 773–792.
[41] Ying, L. (2016). On the Approximation Error of Mean-Field Models. In *Proceedings of the 2016 ACM SIGMETRICS International Conference on Measurement and Modeling of Computer Science* 285–297. ACM.
[42] Ying, L. (2017). Stein's Method for Mean Field Approximations in Light and Heavy Traffic Regimes. *Proc. ACM Meas. Anal. Comput. Syst.* **1** 12:1–12:27. https://doi.org/10.1145/3084449
[43] Zhou, X. and Shroff, N. (2020). A Note on Stein's Method for Heavy-Traffic Analysis. *arXiv preprint arXiv:2003.06454*.