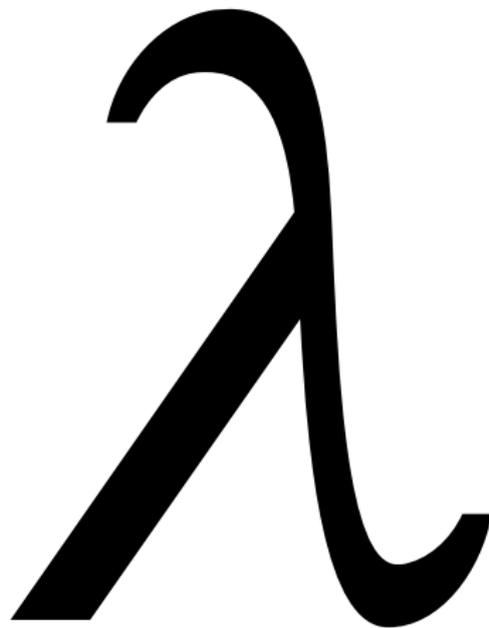


# Recursion Theorem

Ziv Scully

18.504

# P-p-p-plot twist!



# The master plan

- 1 A very  $\lambda$ -calculus appetizer
- 2 Main theorem
- 3 Applications
- 4 Fixed points and diagonalization

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# What is computation?

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- ... following clear procedures step by step (Turing machines).
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- ... complexity that emerges from simple rules (cellular automata).
- ... applying functions to functions to get more functions ( $\lambda$ -calculus).

## Definition of $\lambda$ -calculus

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Additionally, we require that, at the top level, no variable appear outside the scope of a function abstraction that “declares” it.

# Evaluating $\lambda$ -calculus terms

There are two rules for evaluating terms.

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These rules are powerful enough to simulate a Turing machine.

# Hello, factorial!

Let's define the factorial function.

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Problem: we can't have  $F$  refer to itself in  $\lambda$ -calculus.

## Sneakier self-reference

What if we can't use self-reference?

$$F = GG, \text{ where}$$

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Using a self-application  $gg$  for every recursive call is cumbersome. What if we're lazy and want to write  $f$  instead of  $gg$ ?

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$F$  is a “shell” that needs to be filled in by the true factorial function,  $f$ . We can think of  $F$  as having “type”

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If  $f$  computes factorial, then so does  $Ff$ , in which case

$$f = Ff.$$

That is, we necessarily want a fixed point of  $F$ .

## Wishful thinking

Let's wish for a function  $Y$  that finds a fixed point of its input. Namely,  $Y$  should satisfy

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That is, to do recursion, it suffices to find a fixed point of  $F$ .

## Wish really, really sneakily

We know how to do self-reference: make a function accept any reference as an argument, then feed the function to itself. With that inspiration, we wish for some  $G_F$  such that

$$YF = F(YF) = G_F G_F.$$

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## Theorem (Existence of the Y combinator)

*The combinator*

$$Y = \lambda f. G_f G_f = \lambda f. (\lambda g. f(gg))(\lambda g. f(gg))$$

*satisfies  $YF = F(YF)$  for all  $F$ .*

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# Turing machines are computers, too!

Notation:  $[n]$  is the Turing machine  $n$  encodes.

## Theorem (Recursion theorem)

*There exists a total (always halting) computable function  $Y$  such that for all  $F$ , if  $[F]$  is total, then*

$$[Y(F)] = [[F](Y(F))].$$

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$$[Y(F)] = [[F](Y(F))].$$

Put another way: if we consider numbers equivalent if they encode equivalent Turing machines, then  $[F]$  has a fixed point for all  $F$ , and that fixed point is computable from  $F$ .

## Proof of recursion theorem

Notation:  $[n]$  is the Turing machine  $n$  encodes,  $x \mapsto E(x)$  is the procedure that maps  $x$  to expression  $E(x)$ , and  $\langle P \rangle$  is the encoding of procedure  $P$ .

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$Y(F) = Y(F) + 1$ , so the universe explodes.

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We tried to find a fixed point, but we only need a fixed point modulo equivalence of encoded Turing machines.

## Proof of recursion theorem 2: electric boogaloo

We try again with a slightly weaker goal.

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This time, not only can we compute  $G_F$ , but  $[G_F]$  is a total function, so computing  $Y(F) = [G_F](G_F)$  always halts!

## For the skeptical

Just to make sure we have this right, if  $Y(F) = [G_F](G_F)$ , then

$$\begin{aligned}
 [Y(F)](X) &= [[G_F](G_F)](X) \\
 &= [[\langle g \mapsto \langle x \mapsto [[F]([\langle g \rangle](g))](x) \rangle \rangle](G_F)](X) \\
 &= [\langle x \mapsto [[F]([G_F](G_F))](x) \rangle](X) \\
 &= [[F]([G_F](G_F))](X) \\
 &= [[F](Y(F))](X).
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As desired,  $Y(F)$  and  $[F](Y(F))$  encode equivalent Turing machines.

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## Recursion (duh)

Letting  $[F](f) = \langle \dots [f] \dots \rangle$  gives

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In general,  $[F](f) = \langle \dots f \dots \rangle$  gives

$$[Y(F)] = [[F](Y(F))] = \dots Y(F) \dots,$$

which gives us something stronger:  $[Y(F)]$  can access its own source code. Practical application is that, when defining a procedure  $P$ , we can use  $\langle P \rangle$  in the definition.

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This example is simple enough that we can examine  $G_F$  directly.

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$G_F$  says “apply my first argument, decoded, to itself, still encoded”.

$Y(F) = [G_F](G_F)$  decodes  $G_F$  and applies it to a still encoded  $G_F$ .

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This echoes the famous natural-language Quine: “quoted and followed by itself is a Quine” quoted and followed by itself is a Quine.

# Impossibility results

## Theorem (Rice's theorem)

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The procedure

$$Q(x) = \begin{cases} B & P(\langle Q \rangle) \\ A & \text{otherwise} \end{cases}$$

gives a contradiction. □

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## Yet another proof of Cantor's theorem

A surjection  $\hat{a} : \mathbb{N} \rightarrow 2^{\mathbb{N}}$  gives a function  $a : \mathbb{N} \times \mathbb{N} \rightarrow 2$  such that for any  $b : \mathbb{N} \rightarrow 2$ , there is some  $n$  such that  $b = a(n, -)$ , in which case we say that  $b$  is “representable” by  $a$ .

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Let  $\Delta(n) = (n, n)$  and  $\sigma(p) = 1 - p$ .

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A surjection  $\hat{a} : \mathbb{N} \rightarrow 2^{\mathbb{N}}$  gives a function  $a : \mathbb{N} \times \mathbb{N} \rightarrow 2$  such that for any  $b : \mathbb{N} \rightarrow 2$ , there is some  $n$  such that  $b = a(n, -)$ , in which case we say that  $b$  is “representable” by  $a$ .

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By construction, because  $\sigma$  has no fixed points,  $b(n) \neq a(n, n)$  for all  $n$ , which means  $b \neq a(n, -)$  for all  $n$ . Therefore,  $\hat{a}$  is not a surjection.

## Yet another proof the halting problem is undecidable

Suppose  $a : \mathbb{N} \times \mathbb{N} \rightarrow 2$  decides the halting problem. That is,  $a(F, X)$  is 1 if and only if  $[F](X)$  halts.

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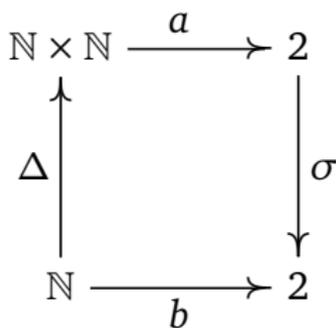
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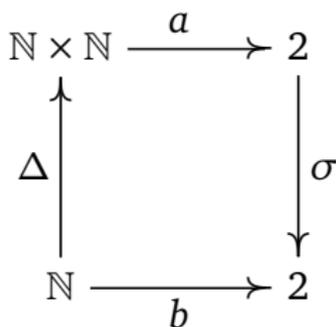
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By construction, because  $\sigma$  has no fixed points,  $b(\langle b \rangle) \neq a(\langle b \rangle, \langle b \rangle)$ . This manifests itself as the usual contradiction:

$$a(\langle b \rangle, \langle b \rangle) = 0 \iff b(\langle b \rangle) \text{ doesn't halt} \iff a(\langle b \rangle, \langle b \rangle) = 1$$

## Yet another cookie-cutter diagonalization proof

The general picture, due originally to Lawvere and nicely explained by Yanofsky (<http://arxiv.org/abs/math/0305282>):

$$\begin{array}{ccc}
 X \times X & \xrightarrow{a} & Y \\
 \Delta \uparrow & & \downarrow \sigma \\
 X & \xrightarrow{b} & Y
 \end{array}$$

$X$  = domain

$\Delta$  = diagonal

$a$  = “application”

$Y$  = “truth values”

$\sigma$  = shuffle

$b$  = “bad”,

where “bad” means not representable as  $a(x, -)$  for some  $x$ .

# Cookies contrapositively cut

$$\begin{array}{ccc}
 X \times X & \xrightarrow{a} & Y \\
 \uparrow \Delta & & \downarrow \sigma \\
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Contrapositively, if  $b$  is representable as  $a(x, -)$  for some  $x$ , then  $\sigma$  must have a fixed point.

## A sketchier, boxier recursion theorem

Fix total computable  $P$ . Let  $a(f, x) = [f](x)$  and  $\sigma([f]) = \sigma([P(f)])$ . (We strategically choose not to worry about how  $\sigma$  is well-defined as part of the composition but isn't on its own.)

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All of  $\Delta$ ,  $a$ , and  $\sigma$  do straightforward computations, so  $b$  is computable and therefore representable as  $a(\langle b \rangle, -)$ .

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All of  $\Delta$ ,  $a$ , and  $\sigma$  do straightforward computations, so  $b$  is computable and therefore representable as  $a(\langle b \rangle, -)$ .

This means  $\sigma$  has a fixed point, or, equivalently,  $P$  has a fixed point modulo equivalence of encoded Turing machines.