$2 \times 2 \cong 4$: An Introduction to Category Theory

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When all you have is the abstractly nonsensical hammer of category theory, everything looks like a commutative diagram.

Introduction

Our goal is to introduce category theory, the study of structures called categories that capture the essence of functions between sets by only assuming the most basic properties of functions hold. By doing this, they become general enough to describe a wide variety of mathematical structures, many of which don't have much at all to do with functions between sets. As we will demonstrate later, several set-specific constructions can be defined in a purely category-theoretic way, unifying many related notions and generalizes them to new contexts where the original construction was not an obvious thing to think about. For instance, products of sets, groups, rings, and topological spaces all fit the same category-theoretic definition, and the same definition describes greatest lower bounds in posets.

The author's introudction to category theory was through a summer course taught by Wofsey [4], and the exposition of the subject given in this paper follows the same general approach as Wofsey's notes while drawing on a wider array of examples. The reader may find a more in depth introduction in [1].

1 The Basics

It has been said that the very beginning is a very good place to start.

Definition 1.1. A *category* $\mathscr{C}$ is

- a collection of *objects*, often written X, Y, or Z;
- a collection of *maps*, also called *morphisms* or *arrows*, each from one object, the *source*, to another, the *target*, often written $f : X \to Y$; and
- a *composition* operation taking $f : X \to Y$ and $g : Y \to Z$ to $gf : X \to Z$

such that

- composition is *associative*, namely $h(gf) = (hg)f$ for any maps f, g, and h with appropriate sources and targets; and
- there is an *identity* map id_X for each object X such that $f\,\mathrm{id}_X = f$ for any $f : X \to Y$ and $\mathrm{id}_X\, g = g$ for any $g : Y \to X$.

Roughly speaking, categories consist of a collection of "set-ish" objects with a collection of "function-ish" maps between objects. In particular, we write composition of maps in the same order function composition is written.

Example 1.2. There is a category of sets, written **Set**. Its objects are sets, its maps are functions, the source and target of a map are respectively the domain and codomain of the function, and its composition is ordinary function composition.

The concerned reader might worry about the size of the collections of objects and maps in **Set**, and indeed, these are both proper classes. For our purposes, all that will matter is that the collection of maps between two particular objects is a set, which will be the case for all the categories we consider.

Example 1.3. There is a category of groups, written **Grp**. Its objects are groups, its maps are group homomorphisms, and its composition operation is ordinary function composition. Proving that **Grp** is a category basically amounts to proving that homomorphisms are closed under composition.

Example 1.4. Many categories have sets with some additional structure for objects and functions that preserve that structure for maps. **Grp** is one such category. Other examples include

- **Ab**, the category of abelian groups, whose maps are group homomorphisms;
- **Ring** and **Rng**, the categories of rings with and without required units, respectively, whose maps are ring homomorphisms;
- **Pos**, the category of posets (partially ordered sets), whose maps are order-preserving functions;
- **Top**, the category of topological spaces, whose maps are continuous functions; and
- $\mathbf{Set}_*$ and $\mathbf{Top}_*$, the categories of pointed sets and pointed topological spaces, respectively, whose maps are functions and continuous functions, respectively, that bring the basepoint of the source to the basepoint of the target.

Example 1.5. Another category, **hTop**, also has topological spaces as its objects but has homotopy classes of continuous functions as its maps. One can define $\mathbf{hTop}_*$ similarly. Both of these categories are important in algebraic topology [3].

Example 1.6. As a first example of a "non-set-ish" category, we can view any poset as a category. Its objects are elements of the poset, and it has a unique map $X \to Y$ if and only if $X \leq Y$. Composing the unique map $X \to Y$ with the unique map $Y \to Z$ to obtain the unique map $X \to Z$ enforces transitivity, and the existence of identity maps enforces reflexivity. (Antisymmetry of the order is not enforced by the definition of categories.)

Example 1.7. One can also view a group as a category. It has a single object and a map from the object to itself for each element of the group with composition given by group multiplication.

We now turn our attention towards redefining familiar set-related concepts in a way that generalizes to categories other than **Set**. One typically defines sets by describing their elements, but the last two examples above show that there is no obvious way to refer to elements of objects in a general category. For instance, it is unclear what an element of 5 in the poset $\mathbb{Z}$ would be. We can make some progress, though: an element of a set X is equivalent to a map $x\colon 1 \to X$ in **Set**, where $1 = \{*\}$ is any one-element set and $x(*) \in X$ is the desired element.

In order to have any hope of defining elements for other categories, we need to define 1 without referring to elements. A first attempt might be the somewhat circular definition that, because the set 1 has a single element, defines 1 to be an object such that there is a unique map $1 \to 1$, namely the identity map. Unfortunately, there is another set with this property: the empty set! In the particular case of **Set**, we can resolve the ambiguity by observing that there is a map $0 \to 1$ (where 0 is the empty set) but no map $1 \to 0$. The definition that ends up being most useful is equivalent in the case of **Set** but stronger in general. (For the following and most other definitions in this paper, we omit formalities like "in a given category $\mathscr{C}$" unless they are needed for disambiguation.)

Definition 1.8. A *terminal* object is an object 1 such that for any object X there is a unique map $!_X : X \to 1$.

Example 1.9. In **Set**, terminal objects are one-element sets. It is clear that there is a unique map from any other set to a one-element set. To see that no other object can be terminal, consider maps from a two-element set 2. There are no maps from 2 the empty set, and there are multiple maps from 2 to any set with at least two elements. Similar reasoning applies to **Pos**, **Top**, and $\mathbf{Top}_*$.

Given this, we could define an element of an object X as a map $1 \to X$ where 1 is some fixed terminal object. Just as all one-element sets are essentially the same, we will soon see that all terminal objects in a category are essentially the same.

Example 1.10. In a poset, the terminal object is the maximum if it exists.

Not all posets have maximum elements, illustrating that not all categories have terminal objects.

Example 1.11. In **Grp**, terminal objects are trivial groups. Every other group clearly has a unique map to the trivial group, and every other group has at least two maps to itself: the identity map and the trivial map, which sends all group elements to the identity element. (These maps coincide for the trivial group.) Similar, but not identical, reasoning applies to **Ab** and **Ring**.

We begin to see the limitation of reasoning using only elements. In **Set**, **Pos**, and **Top**, the categorical definition of elements recovers the typical definition. However, in **Grp**, **Ab**, **Ring**, **Rng**, and $\mathbf{Top}_*$, maps have to preserve an identity element or basepoint, so maps from 1 are too constrained to be of much use.

Recall our earlier discussion about **Set**, in which we tried to define 1 as the object such that there was a unique map $1 \to 1$ but found that there was also a unique map $0 \to 0$. We obtained the correct definition for 1 by strengthening one side, namely by requiring a unique map $X \to 1$ for any X. It turns out that strengthening the other side characterizes the empty set.

Definition 1.12. An *initial* object is an object 0 such that for any object X there is a unique map $?_X : 0 \to X$.

Example 1.13. In **Set**, the only initial object is the empty set. It is perhaps counterintuitive that we are able to define functions from the empty set at all. This is easiest to see by considering the formal definition of a function $f : X \to Y$ as a subset $f \subset X \times Y$ such that for each $x \in X$ there is a unique $y \in Y$ with $(x, y) \in f$. The unique function from the empty set to any other set X is given by $\{\} \subset \{\} \times X$. Similar reasoning applies to **Pos** and **Top**.

Example 1.14. In a poset, the initial object is the minimum if it exists.

Example 1.15. In **Grp**, **Ab**, **Rng**, and $\mathbf{Top}_*$ (but notably not **Ring**), initial objects coincide with terminal objects. For instance, in **Grp**, there is a unique map from a trivial group to any other group because group homomorphisms preserve the identity.

Example 1.16. In **Ring**, there are no maps from a one-element ring to any other ring because no other ring has the additive and multiplicative identities coincide. Instead, one can show that the initial object is $\mathbb{Z}$, which is the “simplest” ring that has both additive and multiplicative identities.

The author was slightly lazy in the last example to say that “the” initial object “is” $\mathbb{Z}$. Though the distinction is inconsequential in practice, we could construct the ring of integers in multiple ways, so **Ring** contains many copies of $\mathbb{Z}$. Of course, they are all isomorphic as rings, and we can express this categorically.

Definition 1.17. An *isomorphism* is a map $f : X \to Y$ such that there exists a map $g : Y \to X$ satisfying $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$. We call two objects X and Y *isomorphic* and write $X \cong Y$ when there is an isomorphism between them.

Proposition 1.18. *Terminal and initial objects are unique up to unique isomorphism. That is, if* 1 *and* $1'$ *are both terminal, then there is a unique isomorphism* $1 \to 1'$*, and similarly for initial objects.*

Proof. Let 1 and $1'$ be terminal. Because $1'$ is terminal, there is a unique map $f : 1 \to 1'$, and, because 1 is terminal, there is a unique map $g : 1' \to 1$. We can compose these to get $gf : 1 \to 1$, but because 1 is terminal, there is a unique map $1 \to 1$, namely id_1, so $gf = \mathrm{id}_1$. Similarly, $fg = \mathrm{id}_{1'}$, so f is an isomorphism. The argument for initial objects is analogous. □

Example 1.19. As mentioned previously, in **Top** and $\mathbf{Top}_*$, terminal objects are one-point spaces by similar reasoning as the **Set** case. In **hTop** and $\mathbf{hTop}_*$, it is not hard to show that terminal objects are contractible spaces, that is, spaces homotopy equivalent to a one-point space. This difference is due to additional isomorphisms in **hTop** and $\mathbf{hTop}_*$. In **Top** and $\mathbf{Top}_*$, isomorphism corresponds to (basepoint-preserving) homeomorphism. In **hTop** and $\mathbf{hTop}_*$, isomorphism corresponds to (basepoint-preserving) homotopy equivalence, a much weaker condition. For instance, $\mathbb{R}^n$ is homotopy equivalent to a one-point space.

2 Categorical Arithmetic

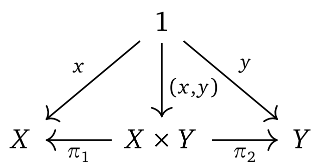
One might describe the product of two sets by writing

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

That is, an element of $X \times Y$ corresponds uniquely to an element of X and an element of Y. How do we express this categorically? We might try directly translating our definition for sets, which would define $X \times Y$ as an object such that there is a bijection between maps $1 \to X \times Y$ and pairs consisting of a map $1 \to X$ and a map $1 \to Y$. In **Set**, this definition picks out an object isomorphic to $X \times Y$, but it won't hold up in other categories for two reasons. First, we have already seen that only considering maps from 1 is inadequate in some categories, especially those in which 1 happens to be initial as well as terminal.

Second, simply stating that there is a bijection doesn't give enough information about the product structure. In **Set**, the product structure is inconsequential because any two sets with the same cardinality are isomorphic, but we can see the problem in other categories. Consider $[0, 1] \times [0, 1]$ as a product of topological spaces. Even in **Top** there is a bijection $[0,1] \to [0,1] \times [0,1]$ given by a space-filling curve, but the inverse of that bijection is not continuous, so it is not an isomorphism in **Top**. Our provisional definition only counts elements, so it cannot distinguish between $[0,1]$ and $[0,1] \times [0,1]$.

We are missing this: given an element $(x,y)\colon 1 \to X \times Y$, we should be able to extract the corresponding $x\colon 1 \to X$ and $y\colon 1 \to Y$ using maps in the category. That is, we should require projection maps $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ such that the diagram below *commutes*, meaning that all compositions of maps along paths going between the same two locations in the diagram are equal.

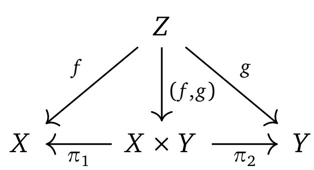


That is, if (x,y) corresponds to elements x and y, we want $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$. This is the essential idea behind the definition of products. All we have to do is generalize to objects other than 1 because maps from 1 aren't as informative in most categories as they are in **Set**.

Definition 2.1. A *product* of two objects X and Y is

- an object $X \times Y$ together with
- two maps, called *projections*, $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$

such that for every object Z with $f\colon Z \to X$ and $g\colon Z \to Y$, there is a unique $(f,g)\colon Z \to X \times Y$ such that the diagram below commutes.



We will often refer to an object alone as a product, in which case the existence of projection maps is implied.

Example 2.2. In **Set**, products are Cartesian products. We can interpret the definition as saying that to define a function $Z \to X \times Y$, we have to define the first coordinate of its output using a function $Z \to X$ and the second coordinate of its output using a function $Z \to Y$.

Example 2.3. In a poset, products are greatest lower bounds. The product definition specifies an element $X \times Y \leq X, Y$ such that if $Z \leq X, Y$ then $Z \leq X \times Y$.

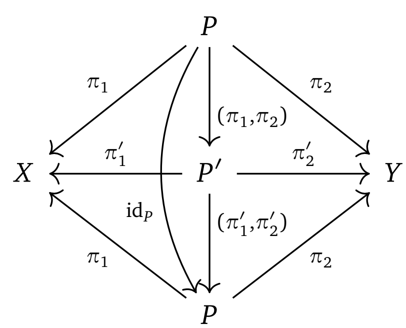
Not all posets have least upper bounds for every pair of elements, illustrating that not all categories products for every pair of objects.

Example 2.4. In **Grp**, **Ab**, **Ring**, **Rng**, products are the direct product of groups or rings.

Example 2.5. In **Top** and $\mathbf{Top}_*$, products are spaces with the product topology. The categorical definition of products captures the fact that the product topology is the coarsest topology for which π_1 and π_2 are continuous. One can define infinite products in essentially the same way we defined products of two objects, and there are at least two sensible-seeming topologies on infinite product spaces in topology, the product topology and the box topology. The categorical product picks the product topology, and that turns out to be the more natural choice, enabling results such as Tychonoff's theorem [2].

Proposition 2.6. *Products are unique up to isomorphism.*

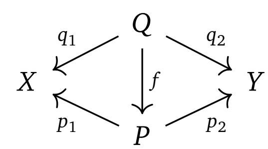
Proof. Let P and P' be two products of X and Y with projections as shown in the diagram below.



Because P' is a product, there is a unique map $(\pi_1, \pi_2)\colon P \to P'$ such that the upper two triangles commute. Similarly, because P is a product, there is a unique map $(\pi'_1, \pi'_2)\colon P' \to P$ such that the lower two triangles commute. Finally, again because P is a product, there is a unique map $P \to P$ that commutes with the outer four maps. It just so happens that id_P commutes with the outer four maps, so $(\pi'_1, \pi'_2)(\pi_1, \pi_2) = \mathrm{id}_P$. Swapping P and P' shows that the reverse composition is also the identity, so $P \cong P'$. □

The proof above closely mimics the proof that terminal objects are unique up to isomorphism. This is not a coincidence.

Example 2.7. Products of X and Y in a category $\mathscr{C}$ are terminal objects of another category, which we call $\mathscr{C}_{X \times Y}$. The objects of $\mathscr{C}_{X \times Y}$ are triples (P, p_1, p_2), where P is an object of $\mathscr{C}$ together with maps $p_1 \colon P \to X$ and $p_2 \colon P \to Y$. The maps $(Q, q_1, q_2) \to (P, p_1, p_2)$ in $\mathscr{C}_{X\times Y}$ are given by maps $f : Q \to P$ in $\mathscr{C}$ such that the following diagram in $\mathscr{C}$ commutes.



To compose maps in $\mathscr{C}_{X\times Y}$, we just compose the corresponding maps in $\mathscr{C}$. Showing that $\mathscr{C}_{X\times Y}$ is a category is not hard and boils down to the fact that composing corresponding maps in $\mathscr{C}$ gives a map that makes the diagram commute, which can be shown by drawing one copy of the diagram on top of another. A terminal object in $\mathscr{C}_{X\times Y}$ is

- an object (P, p_1, p_2) in $\mathscr{C}_{X\times Y}$,
  - or, in $\mathscr{C}$, an object P with maps $p_1 \colon P \to X$ and $p_2 \colon P \to Y$,
- such that for any object in $\mathscr{C}_{X\times Y}$,
  - or, in $\mathscr{C}$, any object Q with maps $q_1 \colon Q \to X$ and $q_2 \colon Q \to Y$,
- there is a unique map to the terminal object,
  - or, in $\mathscr{C}$, a unique map $f : Q \to P$ such that the diagram commutes.

This exactly defines P with projections p_1 and p_2 as a product of X and Y in $\mathscr{C}$.

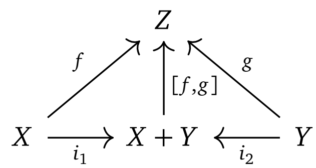
We might ask what the initial object of $\mathscr{C}_{X\times Y}$ is. Unfortunately, the answer is not terribly exciting: if $\mathscr{C}$ has an initial object, then $(0, ?_X, ?_Y)$ is the initial object of $\mathscr{C}_{X\times Y}$. (Recall that $?_X$ is the unique map $0 \to X$ in $\mathscr{C}$.) The trouble is that objects in $\mathscr{C}_{X\times Y}$ come equipped with outgoing maps. Finding the terminal object of $\mathscr{C}_{X\times Y}$ is an object P of $\mathscr{C}$ for which it is easy to construct certain incoming maps, which is useful because we have to supply P with outgoing maps p_1 and p_2 to make it (rather, the triple) an object of $\mathscr{C}_{X\times Y}$, so we end up with information about maps into and out of P. In contrast, the initial object is an object Q of $\mathscr{C}$ for which it is easy to construct certain outgoing maps, which is less interesting because we have to supply Q with outgoing maps q_1 and q_2, so all the information about Q is outgoing maps, which we can make trivial all at once by choosing $Q = 0$.

Recalling the definitions of terminal and initial objects, we see that they are identical save for the direction of the map. It turns out that we get an interesting definition not just by finding an initial object instead of a terminal object but by flipping the directions of all the maps involved and finding the initial object of that category. This is easier to think about in terms of flipping the map directions in Definition 2.8.

Definition 2.8. A *coproduct*, sometimes called a *sum*, of two objects X and Y is

- an object $X + Y$ together with
- two maps, called *inclusions*, $i_1 : X \to X + Y$ and $i_2 : Y \to X + Y$

such that for every object Z with $f : X \to Z$ and $g : Y \to Z$, there is a unique $[f, g] : X + Y \to Z$ such that the diagram below commutes.



We will often refer to an object alone as a coproduct, in which case the existence of inclusion maps is implied.

Example 2.9. In a poset, coproducts are least upper bounds. In this context, flipping map directions in the definition of products flips $\leq$ to $\geq$, which turns greatest lower bounds into least upper bounds.

Just as the unique-up-to-isomorphism proofs for terminal and initial objects are analogous, the proof of the next proposition is essentially the same as the corresponding proof for products.

Proposition 2.10. *Coproducts are unique up to isomorphism.*

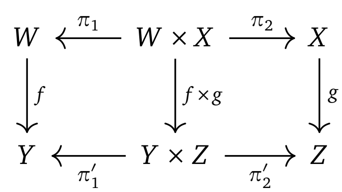
Example 2.11. In **Set**, coproducts are disjoint unions. Because coproducts are unique up to isomorphism, it suffices to show only that disjoint unions satisfy the definition, and the fact that nothing else (that isn't isomorphic) does follows from uniqueness. There are clear inclusion maps into a disjoint union. If we want to specify a function $X + Y \to Z$, it suffices to specify where to map each element coming from X and where to map each element coming from Y. This is exactly given by a pair of functions $f : X \to Z$ and $g : Y \to Z$, and the commutative diagram enforces the fact that the resulting function $X + Y \to Z$ does the same thing as f on the X part and the same thing as g on the Y part.

Example 2.12. In **Grp**, products are free products of groups. In **Ab**, products are free abelian products of groups, which turn out to be the same as direct products. This emphasizes the importance of the context of a category, even when one category is a subcategory of another. Even if X and Y are abelian, the possibility that we might have to map into a nonabelian Z in **Grp** imposes the constraint that elements of X and Y sent into $X + Y$ by the inclusions not commute. In **Ab**, all groups are abelian, so not only must $X + Y$ be abelian if it exists, it is *okay for it to be abelian* because any Z we might have to map into is also abelian.

Example 2.13. In **Top**, coproducts are disjoint unions of topological spaces. In $\mathbf{Top}_*$, coproducts are wedge sums, which are disjoint unions of pointed spaces with the basepoints identified. The arguments are similar to that for **Set**.

We end the section by showing two facts about products that confirm some intuition we might have about them from **Set**.

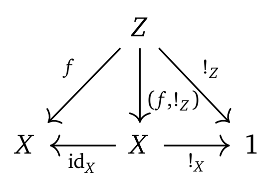
Proposition 2.14. *Given $f : W \to Y$ and $g : X \to Z$, if the products $W \times X$ and $Y \times Z$ exist, then there is a unique map $f \times g : W \times X \to Y \times Z$ such that the following diagram commutes.*



Proof. The diagram above is a thinly-disguised version of the diagram in the definition of products. All we need to do is find compositions that give maps $W \times X \to Y$ and $W \times X \to Z$. Specifically, $f \times g = (f\pi_1, g\pi_2)$. □

Proposition 2.15. *If 1 is a terminal object, then $X \times 1 \cong X$ for any object X.*

Proof.



The above diagram commutes if and only if $(f, !_Z) = f$, so X with projections id_X and $!_X$ satisfies the definition of being a product of X and 1. □

The intuition for this is clear in "set-ish" categories, but it might be surprising at first that it holds in general. Roughly, $X \times Y$ encodes the combined information about maps into X and Y. There is no information to be had about maps into 1 because they are all uniquely determined, so $X \times 1$ only needs to encode information about maps into X, a purpose for which X itself serves adequately.

3 A Powerful Surprise

We informally discussed "flipping map directions" in the previous section in the context of relating terminal objects with initial objects and products with coproducts. This can be made precise.

Definition 3.1. The *opposite* of a category $\mathscr{C}$ is a category $\mathscr{C}^{\mathrm{op}}$ with the same collection of objects, a map $f^{\mathrm{op}}\colon Y \to X$ for each map $f : X \to Y$ in $\mathscr{C}$, and composition given by $f^{\mathrm{op}} g^{\mathrm{op}} = (gf)^{\mathrm{op}}$.

Given a category-theoretic concept (such as a statement or definition) about an arbitrary category $\mathscr{C}$, the *dual* statement or definition is that concept applied to $\mathscr{C}^{\mathrm{op}}$ but interpreted as a statement about $\mathscr{C}$. Concretely, to dualize a concept, one flips all the map directions and reverses the order of every composition. Dual concepts are often named with a "co-" prefix.

Example 3.2. Terminal objects are dual to initial objects, explaining the synonym "coterminal" for initial.

Example 3.3. Products are dual to coproducts. The dual of Proposition 2.15 is that if 0 is a terminal object, then $X + 0 \cong X$ for all objects X.

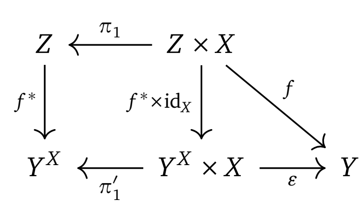
Example 3.4. The dual of the statement $X \times (Y + Z) \cong (X \times Y) + (X \times Z)$ is $X + (Y \times Z) \cong (X + Y) \times (X + Z)$.

The last example tells us that if we could prove that × distributes over + using just the definitions of products and coproducts, we could prove that + distributes over × using just the definitions of products and coproducts. Only one of these is true in **Set**, so there must be some sort of additional structure that exists in **Set** without the dual of that structure also existing. This missing piece takes the form of objects representing a third arithmetic operation: exponentiation! The following definition is a step up in complexity compared to others given so far, but be assured that clarifying examples will quickly follow.

Definition 3.5. An *exponential* of two objects X and Y is

- an object Y^X and
- a *evaluation* map $\varepsilon\colon Y^X \times X \to Y$

such that for any object Z with $f : Z \times X \to Y$, there is a unique $f^*\colon Z \to Y^X$ such that the following diagram commutes, recalling the definition of product maps from Proposition 2.14.

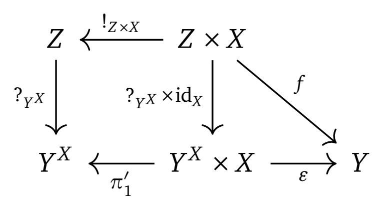


The intuition is that Y^X is an object in a category that represents the arrows $X \to Y$ in that same category. If exponentials for every pair of objects exist, then, in a manner of speaking, the category "knows about its own maps".

Example 3.6. In **Set**, Y^X is the set of functions $X \to Y$. To see this, substitute 1 in for Z and recall that elements of a set are maps into the set from 1. The evaluation map ε is function evaluation: given a function (an element of Y^X) and an argument (an element of X), it yields the result of applying the function to the argument. The definition (with $Z = 1$) enforces that elements of Y^X correspond to functions $1 \times X \cong X \to Y$, and the commutativity of the diagram enforces that the element f^* corresponding to the map f actually represents f and not another function. More generally, if we let Z be arbitrary, the commutative diagram illustrates a correspondence between two-argument functions and one-argument functions that yield one-argument functions given by $f(z, x) = f^*(z)(x)$. Similar things happen in **Top** and **Pos**, though certain conditions on the spaces need to be true for the object in **Top** to exist [2].

Example 3.7. In a boolean algebra (which is a specific type of poset), Y^X represents implication. Furthermore fact, the existence of a complement operation means that the poset is the same as its opposite category. This means boolean algebras have coexponentials, so $+$ should distribute over $\times$ in these categories, and $\vee$ does indeed distribute over $\wedge$ in classical logic.

Example 3.8. In categories like **Grp**, **Ab**, **Rng**, and $\mathbf{Top}_*$ that have an object Z that is both terminal and initial, most exponentials do not exist. The diagram from Definition 3.5 would become the following.



That is, we would have $f^* = ?_{Y^X}$ for any f, which means $f = \varepsilon(?_{Y^X} \times \mathrm{id}_X)$, so there would exist a unique $f : X \times Z \to Y$. Recalling that $X \times Z \cong X$ when Z is terminal, this would mean that there is only one map between any pair of objects for which there exists an exponential. This is obviously not the case for most pairs of objects in these categories.

Recall that for any Z we have a bijection between maps $Z \to X \times Y$ and pairs of maps $Z \to X$ and $Z \to Y$ given in one direction by composition with projections and in the other direction by the definition of the product. However, the existence of the bijection alone is not enough to define $X \times Y$ as the product of X and Y. The bijection is in some way compatible with the categorical structure, namely a certain diagram (the one in the definition of products) commutes.

We have a dual situation for coproducts and even a similar situation for exponentials, where the bijection is between maps $Z \times X \to Y$ and $Z \to Y^X$. With further category theoretic machinery [1] it is possible to precisely define when a bijection is *natural*, that is, when it is compatible with categorical structure in the way those for products, coprodcuts, and exponentials are.

Given two products P and P' of X and Y, for any Z, there is a natural bijection between maps $Z \to P$ and $Z \to P'$ given by composing the natural bijections with pairs of maps $Z \to X$ and $Z \to Y$. Consider the specific case where $Z = P$. The identity map $\mathrm{id}_P \colon P \to P$ corresponds via natural bijection to the pair of projections $\pi_1 \colon P \to X$ and $\pi_2 \colon P \to Y$, which in turn corresponds to $(\pi_1, \pi_2) \colon P \to P'$. This, as we saw in the proof of Proposition 2.6, is an isomorphism. In general, a result called the Yoneda lemma [4] tells us the following: if for any Z there is a natural bijection between maps $Z \to X$ and maps $Z \to Y$, then there is an isomorphism $X \to Y$. In particular, as in our product example, the isomorphism corresponds to id_X via the natural bijection. With this, we are a few short lemmas away from our main theorem.

Lemma 3.9. *Products and coproducts are commutative and associative, so finite products and coproducts of more than two objects are well-defined.*

Proof. Commutativity follows immediately from the symmetry of the definitions. Associativity is pretty easily shown directly in a similar proof to that of Proposition 2.6. Even more simply, we can give a natural bijection, as we do here for the product case, from maps into $X \times Y \times Z$ parenthesized arbitrarily to triples of maps into each of X, Y, and Z. Composing one such bijection with the inverse of another gives a natural bijection between maps into two different parenthesizations. □

Lemma 3.10. *In a category with products, coproducts, and exponentials of every pair of objects,* $X \times (Y + Z) \cong (X \times Y) + (X \times Z)$.

Proof. As discussed, it suffices to give a natural bijection between maps out of the objects on each side. All of the following are in natural bijection:

- maps $(Y + Z) \times X \to W$,
- maps $Y + Z \to W^X$, by the definition of exponentials,
- pairs of maps $Y \to W^X, Z \to W^X$, by the definition of coproducts,
- pairs of maps $Y \times X \to W, Z \times X \to W$, by exponentials again, and
- maps $(Y \times X) + (Z \times X) \to W$, by coproducts again. □

Theorem 3.11. $(1 + 1) \times (1 + 1) \cong 1 + 1 + 1 + 1$.

Proof. We leave the proof as an exercise to the particularly advanced reader. □

References

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